Beating the Correlation Breakdown:
Robust Inference, Flexible Scenarios, and Stress Testing for Financial Portfolios

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Disclaimer: The views presented in this monograph are those of the sole author, JD Opdyke, and do not necessarily reflect those of particular institutions.

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JD came to SCGAM from the Abu Dhabi Investment Authority (ADIA) where he was quant lead of portfolio oversight in a rapidly growing alpha factory. Prior to that he held senior management roles at Allstate and GE Capital, and previously served as a senior quantitative consultant at a number of global banking institutions. JD has built from scratch and led several senior quant modeling teams, published over a dozen peer reviewed journal papers and book chapters, several of which were voted 'Paper of the Year' by panels of experts, and is a frequently invited speaker and presenter at top quant investing and risk conferences globally.

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Introduction

We live in a multivariate world, and effective modeling of financial portfolios, including their construction, allocation, forecasting, and risk analysis, simply is not possible without explicitly modeling the dependence structure of their assets. Dependence structure can drive portfolio results more than many other parameters in investment and risk models – sometimes even more than their combined effects.

> "Correlation is one of the most important, if not the most important, risk factor in finance, driving everything ... however, a unified and generally accepted correlation risk management framework does not yet exist" (Packham & Woebbeking, 2023, p.1).

Despite this widely acknowledged yet unacceptable state of affairs, the literature provides relatively little to define the finite-sample distributions of commonly applied dependence measures, like (Pearson's) correlation, in useable and useful ways under challenging, real-world financial data conditions.[1] Yet this is exactly what is needed to make valid inferences about their estimates, and to use these inferences for a myriad of essential purposes, such as hypothesis testing, dynamic monitoring, realistic and granular scenario and reverse scenario analyses, as well as mitigating the effects of correlation breakdowns during, and preferably before, market upheavals (which is when we need valid inferences the most).

On the one hand the dearth of "real-world effective" methodology in this setting, which again simply boils down to defining the finite-sample distributions of the relevant dependence measures, is somewhat surprising. Financial markets certainly have seen more extreme downturns in recent decades than many would have expected ex ante (e.g. Black Monday (1987), Tech Bubble (2000), Housing Bubble (2008), Covid (2020)), during which correlation breakdowns have been well documented, their very material effects measured and assessed (see for example Feng & Zeng, 2022, and Packham & Woebbeking, 2023), and the importance of mitigation efforts widely discussed, considered, and acted upon (see Greenspan, 1999; BIS, 2011a; and EBA-CRR, 2013). What's more, practitioners, academics, and regulators have a long history of bringing analytic and probabilistic rigor to bear when analyzing and estimating *the other* parameters of our portfolio risk and investment models. There is no shortage of empirical research defining, for example, various estimators of the tail indices of a portfolio's marginal distributions, and deriving their associated p-values, confidence intervals, and statistical power and level. When rigorously and properly estimated, these tools are highly actionable, providing invaluable guidance in decision-making and mitigation efforts. But what do we find when we look for those same tools, say, for an entire matrix of pairwise Kendall's tau values, to make decisions based on answers to questions such as, "Has this Kendall's matrix shifted in the past week? What is the probability of observing the movement we observed, given that our baseline estimate is true? Does this meet our probabilistically defined threshold

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[1] I take 'real-world' financial returns data to be multivariate with marginal distributions that can vary notably from each other, and change in time, in their degrees of heavy-tailedness, serial correlation, asymmetry, and (non-)stationarity. These obviously are not the only defining characteristics of such data, but from a distributional and inferential perspective, they remain some of the most challenging, especially when occurring concurrently as they do in non-textbook settings.

for a 'breakdown'?" or "Are the Kendall's matrices from these two sectors different only due to sample variation? Or do they represent fundamentally distinct dependence structures?" or "What are the two (upper and lower) Kendall's matrices that capture 95% of the conditional sample variation in this setting? How far beyond these bounds, probabilistically, does each Kendall's matrix that we've defined for each of our scenarios lie? Given our distributions of losses/returns, does a tail dependence matrix better capture what we are trying to measure here, and can we conduct a ceteris paribus analysis, using the exact same distribution-defining methodology, to compare the statistical power of these two dependence measures under the various relevant data conditions?" If we require the p-values and confidence intervals and rigorous, probabilistic answers to these questions to be valid under challenging, real-world financial data conditions, the current literature provides relatively little.

On the other hand, the possible explanations for this disparity in useable and useful methodology are not entirely unreasonable. First, depending on how broadly we define it, this arguably is a harder problem than, say, that related to univariate tail indices, or many of the challenges related to modeling the other parameters of investment and risk portfolios. Even though each cell value of the dependence matrix is a bivariate association, we are measuring all the pairwise associations in the portfolio simultaneously, and the values of the cells are, in non-trivial cases, all interrelated, making this a complex, multivariate problem. Immutable mathematical requirements such as positive definiteness arise, and make deriving and simulating the distribution of the all-pairwise matrix a non-trivial task. This is especially true if we want to require, as we should, that the p-values and confidence intervals of each and every cell are consistent with those of the entire matrix. Additionally, requiring that the finite sample distribution of the matrix (which makes possible the calculation of the p-values and confidence intervals) remain valid under challenging, real-world financial data conditions adds significantly to the nontrivial nature of the problem. Distributions of dependence measures are more readily derived when we can assume that returns are, say, multivariate normal, or at least independent and identically distributed (iid). It is another matter entirely when the portfolio's marginal distributions vary notably from each other, while also changing over time in their degrees of heavy-tailedness, serial correlation, asymmetry, and (non-)stationarity. Yet this is exactly the empirical challenge of actual financial portfolios. In fairness, the literature *does* provide many solutions under mathematically convenience conditions, which more narrowly define and restrict both in the distributional characteristics of the underlying returns data as well as the assumptions made regarding the values of the all-pairwise matrix. But these often unrealistic assumptions limit practical, real-world application, which is exactly the motivation for this monograph.

Another complicating factor is the requirement that the method defining the finite sample distribution of the relevant dependence measures is the same across all those in practical usage: in this case, all those dependence measures for which the all-pairwise matrix is positive definite.[2] This arguably covers all that

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[2] Note that "positive definite" throughout this monograph refers to the dependence measure calculated on the matrix of all pairwise associations in the portfolio, that is, calculated on a bivariate basis. Some of the dependence measures addressed in this monograph (e.g. Szekely's correlation, variants of Chatterjee's, and others) can be applied on a multivariate basis (sometimes even in arbitrary dimensions), for example, to test the hypothesis of multivariate independence. But "positive definite" herein is not applied in this sense (see for example Cardin, 2009), and I explain below some of the reasons for using

could conceivably be used and be useful in the financial setting. This universality is certainly desirable, but it also increases the challenge of deriving the methodology. However, this actually is a crucial requirement as it provides the ability to conduct all-else-equal analyses comparing the performance of different dependence measures under controlled conditions: we can be certain that material and statistically significant differences in results are due to the dependence measures themselves, rather than distinct methodologies we typically would have to use (based on the current literature) to define their distributions. Yet nothing in the extant literature provides this broad ceteris paribus capability.

Finally, one of the major uses of dependence measures and their all-pairwise matrices is in defining scenarios and reverse scenarios.[3] These remain central and critical for all manner of risk analyses, and fully flexible scenarios require the ability to 'freeze' any set of selected cells of the all-pairwise matrix while allowing others to vary. For example, many of the pairwise cells that will change dramatically, in both direction and magnitude, under a Covid-like scenario will be completely unaffected under a housing bubble (see Feng & Zeng, 2022, and Pramanik, 2024), and scenario analytics must be able to validly define the finite sample distribution of the all-pairwise matrix under both types of scenario-restricted conditions. However, no existing method allows for this without inadvertently affecting the other 'peripheral' cells of the all-pairwise matrix (see Ng et al. (2014) and Yu et al. (2014)), and this can dramatically distort the distribution of the matrix, rendering the associated inferences for the scenario useless at best, and dangerously misleading at worst. Granular flexibility in scenario definition, at the level of the pairwise cells, and *the valid distribution of the associated, scenario-restricted all-pairwise matrix*, is a necessity if we are to accurately capture the fundamentally different nature of disparate correlation breakdowns, and accurately assess, forecast, and mitigate their impacts.

So perhaps it is not so surprising that we have comparatively little in the extant literature in the way of real-world solutions to this problem. For purposes of this monograph, I define the problem statement as follows: to define the finite sample distributions of all positive definite measures of dependence structure, robustly under real-world data conditions, that remain valid regardless of the estimators used, and even if the co-movement of selected pairs of variables is 'frozen', i.e. scenario-restricted. In contrast to this problem statement, to date financial portfolio analysis in practice very often relies on ad hoc, largely qualitative, and 'judgmental' approaches to specifying and utilizing dependence structure. When quantitative approaches are used, their application relies on a literature largely restricted to providing solutions valid only in narrowly defined cases and/or requiring unrealistic but mathematically convenient assumptions for which either i. the distributions derived are only asymptotically valid (i.e. assume infinitely large sample sizes), or ii. they require very restrictive and/or unrealistic assumptions about the marginal returns distributions of the portfolio (e.g. that they are multivariate Gaussian, or elliptical; or

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the dependence framework of all pairwise associations, which is highly flexible, and allows for more precise attribution and intervention analyses.

[3] Scenarios typically are designed to answer questions of the type, "What loss is associated with, say, the 99.5%tile of the loss distribution?" while reverse scenarios answer questions of the type, "What percentile of the loss distribution produces a loss of $X?" The dollar amounts referenced in the latter typically are associated with specific extreme or catastrophic events, such as insolvency or the failure of a major business line or geography.

even that they are independent and identically distributed ("iid"), or all symmetric, or all stationary, or not serially correlated, etc.), or iii. they require very restrictive and/or unrealistic assumptions about the values of the dependence measures themselves (e.g. the cells are all zeros, or all have the same value, or follow very discrete and limited block structures), or iv. they estimate the all-pairwise matrix in ways that do not guarantee its positive definiteness, or violate other fundamental mathematical requirements (e.g. unit diagonals), or v. most typically, they require multiple of these restrictive and/or unrealistic assumptions combined. Many of these more narrow solutions are mathematically elegant, but our goal herein is to obtain an actual solution that works and remains inferentially valid for messy, challenging, real-world financial portfolios. The new and straightforward method developed herein – Nonparametric Angles-based Correlation ("NAbC") – is defined by eight critically important characteristics listed below that, satisfied simultaneously, distinguish it from any other approach in the literature. Yet its foundations rest squarely on very well established results in the relevant and closely related literatures.

1. NAbC remains valid under challenging, real-world data conditions, with marginal asset distributions characterized by notably different and varying degrees of serial correlation, (non-)stationarity, heavy-tailedness, and asymmetry

2. NAbC can be applied to ANY positive definite dependence measure, including but not limited to the foundational Pearson's product moment correlation matrix (Pearson, 1895), rank-based measures like Kendall's Tau (Kendall, 1938) and Spearman's Rho (Spearman, 1904), measures designed to capture highly non-linear and/or cyclical dependence such as the tail dependence matrix (see Embrechts, Hofert, and Wang, 2016, and Shyamalkumar and Tao, 2020), Chatterjee's correlation (Chatterjee, 2021) and its variants (Pascual-Marqui et al., 2024), the improved Chatterjee's correlation (Xia et al., 2024), Lancaster's correlation(s) (Holzmann and Klar, 2024), and Szekely's distance correlation (Szekely, Rizzo, and Bakirov, 2007) and its variants (such as Sejdinovic et al., 2013, and Gao and Li, 2024)

3. NAbC remains "estimator agnostic," that is, valid regardless of the sample-based estimator used to estimate any of the above-mentioned dependence measures

4. NAbC provides valid confidence intervals and p-values at both the matrix level and the pairwise cell level, with analytic consistency between these two levels (i.e. the confidence intervals for all the cells define that of the entire matrix, and the same is true for the p-values; this effectively facilitates, and in many cases makes possible, granular and targeted attribution analyses)

5. NAbC provides valid confidence intervals and p-values not only for one-sample tests against matrices of fixed, assumed 'true' values, but also for two-sample tests comparing two matrices, so that we can assess inferentially whether dependence structures truly are different, for example, across different sectors or segments of our businesses.

6. NAbC provides a one-to-one quantile function, translating a matrix of all the cells' cumulative distribution function (cdf) values to a (unique) correlation/dependence measure matrix, and back again, enabling precision in reverse scenarios and stress testing, as well as informed and targeted 'what if' analyses

7. all the above results remain valid even when selected cells in the matrix are 'frozen' for a given scenario or stress test – that is, unaffected by the scenario – thus enabling flexible, granular, and inferentially valid realistic scenarios
8. NAbC remains valid not just asymptotically, i.e. for sample sizes presumed to be infinitely large, but rather, for the specific sample sizes we have in reality (for full-rank matrices with n>p)[4], enabling inferentially reliable application in actual, real-world, non-textbook settings

The alternative to a method satisfying the eight objectives above, simultaneously, is to use a piecemeal, incomplete patchwork of disparate derivations of distributions, some asymptotic, some not, valid under typically restrictive and unrealistic data conditions for only a few of the widely used dependence measures: in the end, the patchwork limits, and arguably all but prohibits, reliable comparative analyses which, if attempted, in reality become more confounding than elucidating. This is exacerbated by the unwieldy, opaque, highly complex, and difficult-to-implement nature of many of these solutions.

NAbC circumvents all of these problems with a single, unified, and straightforward method that, compared to its more limited and narrowly defined competitors, simultaneously and dramatically increases i. robustness, ii. scenario flexibility, iii. accuracy in attribution analyses, and iv. targeted precision in 'what if' intervention analyses, all while enabling v. ceteris paribus analyses across dependence measures. What's more, in satisfying the eight objectives listed above, NAbC also satisfies the original motivating factor for its development: it provides the same level of probabilistic rigor in the objective, quantitative analysis of portfolio dependence structure as has been applied to the other parameters in our investment and risk portfolio models.

Importantly, note that the original statement in this Introduction, "effective modeling of financial portfolios, including their construction, allocation, forecasting, and risk analysis, simply is not possible without explicitly modeling the dependence structure of their assets." applies to *all* frameworks for portfolio analysis, even those that may not always make explicit their estimation of, or their reliance on, dependence structure. For example, some path dependent approaches generate distributions of portfolio results based in large part, or even primarily, on (usually subjectively defined) probabilities associated with various scenarios, without explicitly defining dependence structure. But such approaches still make many *implicit* assumptions regarding dependence structure, such as that it does not change from one period to the next, or that it does not change under one scenario versus another, or that, even if (Pearson's) correlations may be controlled via 'views' specified in the model, other measures of dependence, such as tail dependence, are not unwittingly changed from one period to another (even if this is unlikely when 'views' on volatilities are changed). Whether implicit, indirectly explicit via 'views' on other parameters, or explicit, all such assumptions about dependence structure will affect simulated results, and they always should be made fully explicit in any model (see Meucci, 2010b, and Vorobets, 2025, for examples), even if only for ex post testing using NAbC to ensure that the effects of (possibly changing) dependence structure are not (unknowingly) confounding results.

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[4] Recall that this condition is required for the all-pairwise matrix to be positive definite.

But beyond and in addition to simply avoiding confounding, NAbC provides such models with statistical control and inferential validity when specifying dependence matrix values based on scenario 'views'. As opposed to ad hoc or judgement-based matrix values, the 'view' of an extreme correlation/dependence matrix should be defined probabilistically, based directly on its finite sample distribution, which NAbC provides. For example, the 'view' of a correlation matrix corresponding to an extreme scenario, when used as an input to a path dependent simulation, should be a percentile (say, 99%tile) of the distribution of the all-pairwise matrix, as provided by NAbC's quantile function. All that NAbC needs to define the values of this matrix is the data generating mechanism and the null hypothesis, i.e. the baseline values of the matrix. Conversely, NAbC also can provide the cdf value (percentile) of an all-pairwise matrix whose values are specified for a specific scenario, thus ensuring that it is sufficiently 'extreme,' or not too 'extreme,' (e.g. is it the 80%tile? Or the 99.999%tile?) for the scenario being tested. NAbC provides both: the matrix corresponding to a specified percentile, and the percentile corresponding to a specified matrix. Only in this way does a 'view' on dependence structure retain objective meaning regarding its relative size, not to mention its inferential validity, as opposed to being informed by qualitative, subjective judgements or ad hoc procedures.

I conclude this Introduction with a request of my valued readers: at various stages in this monograph you will recognize that other papers I am discussing and presenting will provide solutions, often very good ones, to pieces of the problem statement I define above. However, they all have notable if not mortal shortcomings when all eight of the objectives I have listed need to be achieved simultaneously, which is what I have found is required for the applied solution to this problem in non-textbook financial settings. I always try to explain, as proximate as possible in the text, where and why these limitations restrict the scope of application of a particular, more narrow solution, so I would gently ask the reader to suspend disbelief long enough to allow me to present these before deciding I have missed an important, existing solution in the literature. However, if that does turn out to be the case in any specific instance, I stand eager to learn of and correct the oversight.

Background

The primary objective of this monograph is to develop and implement a new and straightforward method – Nonparametric Angles-based Correlation ("NAbC") – for defining the finite-sample distributions of a very wide range of dependence measures for financial portfolio analysis. As described above, these include ANY that are positive definite, such as the foundational Pearson's product moment correlation matrix (Pearson, 1895), rank-based measures like Kendall's Tau (Kendall, 1938) and Spearman's Rho (Spearman, 1904), as well as measures designed to capture highly non-linear and/or cyclical dependence such as the tail dependence matrix (see Embrechts, Hofert, and Wang, 2016, and Shyamalkumar and Tao, 2020), Chatterjee's correlation (Chatterjee, 2021), Lancaster's correlation (Holzmann and Klar, 2024), and Szekely's distance correlation (Szekely, Rizzo, and Bakirov, 2007) and their many variants (such as Sejdinovic et al., 2013, and Gao and Li, 2024).

Without an estimator's finite sample distribution, valid inferences simply cannot be made about its estimates from actual data samples. While much has been accomplished regarding distributional results and inferential capabilities of estimators of these dependence measures for the bivariate case, this is different from application to an entire (multivariate) portfolio, where estimation of the all-pairwise matrix imposes additional constraints and complications (e.g. the requirement of positive definiteness). This is where NAbC comes in. Motivation for NAbC's development has been enabling such inferential analyses in real-world settings, under real world data conditions, for actual financial portfolios (as opposed to textbook settings, or strictly the bivariate case). Fortunately, the development of dependence measures like the ones listed above not only is foundational for modern statistics, but also has been a very active area of research recently. So NAbC has no shortage of ready-made, well tested, useful and usable dependence measures to rely on in its inferential application. I first review these dependence measures in this chapter before demonstrating how NAbC enhances and in many cases, enables their rigorous, inferential application to financial portfolios.

Types of Dependence Measures

Measures of association, otherwise known as dependence measures, are as old as modern statistics itself (see Pearson, 1895). They provide a quantitative assessment of how variables move together or in opposite directions over time. I address their relation to causal mechanisms in later chapters, and merely note here that they remain distinct from what are often called 'metrics' or 'distance metrics,' even though the two are sometimes confused.[5]

Monotonic Dependence Measures

The oldest and most widely used and known dependence measure is Pearson's product moment correlation (see Pearson, 1895), which is what is usually referenced when "correlation" is mentioned. Taking two variables, say, the financial returns of two assets X and Y, Pearson's measures how often and to what degree they deviate from their respective sample means in the same or in opposite directions, as

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[5] Even though 'metrics' or 'distance metrics' often are built directly on dependence measures, they typically do not share many of their characteristics (e.g. their spaces are not necessarily positive definite (see Alpay & Mayats-Alpay, 2023; and Meckes, 2013)) In finance they often are used non-inferentially and mechanistically in hierarchical portfolio construction models (see Tumminello et al., 2005; and Dom et al., 2024) where they have received mixed reviews (see Aznar, 2023; Cota, 2019; and Ciciretti & Pallotta, 2023), especially under correlation breakdowns (see Marti et al., 2021). However, a recent and ingenious paper by Cotton (2024) provides the first methodological basis for their (partial and conditional) usage for portfolio construction. Still, as they stand, they are not designed to answer the inferential questions posed herein. In fact, I show in later chapters how NAbC provides a generalized entropy that has many useful advantages over an entire class of metrics most commonly used in this setting, called 'norms.'

shown in (1) below.[6]

$$(1) \quad r_{X,Y} = \frac{\displaystyle\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{j=1}^{n}X_j\right)\left(Y_i - \frac{1}{n}\sum_{j=1}^{n}Y_j\right)\Big/(n-1)}{\sqrt{\displaystyle\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{j=1}^{n}X_j\right)^2\Big/(n-1)}\sqrt{\displaystyle\sum_{i=1}^{n}\left(Y_i - \frac{1}{n}\sum_{j=1}^{n}Y_j\right)^2\Big/(n-1)}} = \frac{Cov(X,Y)}{s_X s_Y}$$

The numerator is the (sample) covariance of X and Y, and the denominator – the product of the (sample) standard deviations of X and Y – has the effect of scaling the (sample) covariance to a (maximum) range of -1 to 1.[7] So Pearson's is just the scaled covariance between X and Y.

Another of the most commonly used dependence measures is Spearman's Rho (see Spearman, 1904), which is exactly the same formula as Pearson's but instead of using the values of X and Y, their ranks are used instead:

$$(2a) \quad sr_{X,Y} = \frac{\displaystyle\sum_{i=1}^{n}\left(R_{X_i} - \frac{1}{n}\sum_{j=1}^{n}R_{X_j}\right)\left(R_{Y_i} - \frac{1}{n}\sum_{j=1}^{n}R_{Y_j}\right)\Big/(n-1)}{\sqrt{\displaystyle\sum_{i=1}^{n}\left(R_{X_i} - \frac{1}{n}\sum_{j=1}^{n}R_{X_j}\right)^2\Big/(n-1)}\sqrt{\displaystyle\sum_{i=1}^{n}\left(R_{Y_i} - \frac{1}{n}\sum_{j=1}^{n}R_{Y_j}\right)^2\Big/(n-1)}}$$

If there are no ties in the data, (2a) can be shortened to

$$(2b) \quad sr_{X,Y} = 1 - \frac{6\displaystyle\sum_{i=1}^{n}\left(R_{X_i} - R_{Y_i}\right)^2}{n^3 - n} \quad \text{(see Zar, 1999)}$$

Using ranks can make Spearman's less sensitive than Pearson's to extreme data values under some data conditions, just like another rank-based dependence measure, Kendall's Tau.

Also called a measure of concordance, Kendall's Tau (see Kendall, 1938) is the sum of all pairwise comparisons of every data point of X and Y, divided by the total number of pairs. The pairwise comparisons are given values of 1, 0, or -1, respectively, if both from one period to another are in increasing/decreasing order, if the values from both periods are tied for either of the assets, or if both assets are NOT in increasing/decreasing order; it thus gives the number of pairs in concordance minus the number in discordance relative to the total number of pairs, as shown below.

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[6] Importantly, all formulae of estimators herein, unless otherwise noted, refer to those based on sample data, as opposed to those based on an entire population of data.

[7] Note that this range can be tighter under specific circumstances, such as for equicorrelation matrices where $\left[-1/(p-1)\right] \le r \le 1, \ p = \dim(r)$.

$$(3a) \quad \tau(X,Y) = \frac{\text{\#concordant pairs} - \text{\#discordant pairs}}{\text{total \# pairs}} = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \text{sgn}(x_i - x_j) \text{sgn}(y_i - y_j)$$

where $\text{sgn}(z) = 1$ if $z > 0$, $\text{sgn}(z) = -1$ if $z < 0$, $\text{sgn}(z) = 0$ if $z = 0$, for both N and n

However, ties in the values of either of the pairs, $(x_i \text{ and } x_j)$ or $(y_i \text{ and } y_j)$, will restrict the range from achieving -1 or +1, even under otherwise perfect discordance or concordance, respectively, so a commonly used variant of Kendall's Tau that avoids this drawback when ties exist is:

$$(3b) \quad \tau_b(X,Y) = \frac{1}{\sqrt{(n_0 - n_1)(n_0 - n_2)}} \left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \text{sgn}(x_i - x_j) \text{sgn}(y_i - y_j) \right]$$

where $n_0 = n(n-1)/2$; $n_1 = \sum_{i=1}^{i \text{ grps ties}} t_i(t_i - 1)/2$; $n_2 = \sum_{j=1}^{j \text{ grps ties}} u_j(u_j - 1)/2$;

$t_i = \text{\# ties in i-th group of } x$; $u_j = \text{\# ties in j-th group of } y$

The "big 3" dependence measures – Pearson's, Kendall's, and Spearman's – are by far the most widely used in practice.[8] Although widely held myths persist regarding Pearson's as a measure strictly of *linear* monotonic relationships (see van den Heuvel & Zhan, 2022), all three measure 1. monotonic association (i.e. the direction of the association, positive or negative, does not change within the covered time period) that is 2. symmetric, or non-directional in the variable order (i.e. the measured dependence of X on Y is the same as that of Y on X). It is important to recall here that as measures of monotonic dependence, values of zero generally do not necessarily imply independence between X and Y,[9] but independence between X and Y does imply values of zero for the big 3.[10] Many of the dependence measures treated below avoid this conceptual shortcoming.

The properties of the big 3 have been studied extensively in the literature, but real gaps remain. Our interest in this monograph lies not just in a single bivariate relationship between X and Y, but rather, in all pairwise relationships of all assets in a portfolio, simultaneously: X may be strongly, positively associated with Y, which also may be positively associated with Z, which also may be negatively associated with A, B, and C, while B and C may be modestly but negatively associated with X again! So we have a matrix of dependence measure values with rows and columns identifying the pairwise relationships between all

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[8] Others include Hoeffding's D (see Hoeffding,1948), Blomqvist's coefficient (see Blomqvist, 1950), and Gini's gamma (see Gini 1914; and Genest et al., 2010).

[9] However, an exception occurs when data is distributed as bivariate normal, in which a Pearson's value of zero *does* indicate independence.

[10] This is easy to visualize with a non-monotonic relationship like $y = x^2 + \varepsilon$, which on average will yield big 3 values close to zero. But the relationship is non-linear and u-shaped, which most certainly is not one of independence.

the asset pairs, as shown in (4) for assets 1, 2, 3, and 4. I refer to this matrix herein as the all-pairwise matrix.

$$R = \begin{bmatrix} 1 & r_{1,2} & r_{1,3} & r_{1,4} \\ r_{2,1} & 1 & r_{2,3} & r_{2,4} \\ r_{3,1} & r_{3,2} & 1 & r_{3,4} \\ r_{4,1} & r_{4,2} & r_{4,3} & 1 \end{bmatrix}$$

(4)

Some of the characteristics of this matrix, for all of the big 3 and many of the other measures presented below, include:

i. Symmetry: $r_{i,j} = r_{j,i}$

ii. Unit diagonal entries: $r_{i=j} = 1$

iii. Bounded non-diagonal entries, with maximum range of: $-1 \le r_{i,j} \le 1$

iv. The matrix is positive definite,[11] i.e. all eigenvalues $\lambda_i > 0$

For completeness and for reference throughout this monograph, I define eigenvalues here:

If there exists a nonzero vector v such that $Rv = \lambda v$ then λ is an eigenvalue of R and v is its corresponding eigenvector. λ and v can be obtained by solving $\det(\lambda I - R) = 0,$ then $\det(\lambda I - R)v = 0,$ where I is the identity matrix and det is the determinant. The eigenvalue can be thought of as the magnitude of the (portfolio) variability in the direction of the eigenvector. With actual, real-world financial data (i.e. values that are not imaginary or complex), this variability can never be negative,[12] so numeric computational issues aside,[13] proper measures of dependence must be positive definite.[14]

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[11] See Saboto et al. (2007) for proofs of the (semi)positive definiteness of Pearson's, Kendall's, and Spearman's. Semi-positive definiteness includes the case of eigenvalues exactly equal to zero, which I largely ignore herein as a border case relevant mainly for textbook examples.

[12] This can be seen most easily when the covariance (or equivalently, Pearson's correlation) is the dependence measure used: the covariance is the (expected value of a) sum of squared real numbers (as no imaginary or complex values are observed in financial returns. Because a squared, real number greater than zero is always greater than zero, their sum can never be negative.

[13] Numerical calculations can sometimes render estimates of specific eigenvalues slightly negative, but the NAbC method proposed herein in later chapters specifically is designed to be more robust to such numerical errors than the more common, limited approaches related to eigen decompositions in the extant literature.

[14] If any $\lambda = 0$ the matrix is said to be positive semi-definite, although herein this is treated as a textbook border case as all relevant financial returns would have to be exactly zero for the eigenvalue to be zero.

Positive definiteness is proven mathematically in later chapters for all widely used dependence measures in this setting, and it remains crucial for a number of important reasons discussed throughout this monograph. But the main point here is that we need to understand the characteristics of the estimators we use to estimate the values of the all-pairwise matrix, based on our sample of financial returns data. Why? Because we want to be able to make *inferences* from these sample-based values to answer critical questions about the behavior of the portfolio, and specifically its dependence matrix. For example, Q: has the dependence matrix changed from our established baseline, or from period 1 to period 2? (A: one-sample or two-sample matrix-level p-value, respectively) If so, Q: which pairwise relationships primarily are driving this conclusion? (A: attribution analyses using cell-level p-values) Q: How likely is scenario "E" given our current baseline dependence matrix of "F"? (A: one-sample matrix-level p-value) Q: What are the two matrices that correspond to the 95% confidence interval for our current matrix "C"? (A: matrix level confidence intervals) Q: Do the matrices associated with our scenarios and stress tests fall outside these two matrices? If not, do we need to revisit their construction? Q: What is the probability of observing the matrix associated with reverse scenario "X"? (A: matrix level quantile and p-value) Q: If pairwise cells A, B, and C remain unaffected by Scenario Q, how much do cells X, Y, and Z have to change for dependence measure #3 to be able to detect the change? (A: scenario-restricted cell-level and matrix-level p-values) What is the answer to this question for dependence measure #4? What are the implications of this comparison for detecting correlation breakdowns? Are the power and the robustness and the average run length of dependence measure #3, when used in the appropriate statistical process control chart, better than #4? Or are there tradeoffs between the two dependence measures? Do answers to these questions depend on the (sample-based) estimators used for each, #3 and #4? (A: matrix and cell-level p-values and confidence intervals, respectively).

The list of relevant, financially material inferential analyses here is endless, but one thing is necessary to be able to answer any of these questions with probabilistic rigor: we *must* have the finite-sample distribution of whatever estimator we are employing to estimate the values of the all-pairwise matrix. Conceptually this is the same as using a "bell" curve (a.k.a. a normal or Gaussian distribution) to answer how likely it would be to observe, for a single variable, a sample mean at least as large as what we actually observe in practice:[15] in our case, with the all-pairwise dependence matrix, the question could be identical except that the distribution simply is multivariate (i.e. many variables) as opposed to univariate (one variable), and in almost all cases the distribution would not be Gaussian.

However, the extant literature provides relatively little to define the multivariate, finite sample distributions of these dependence measures under real-world data conditions, let alone the use of a common method across all relevant dependence measures. Either the distributions derived are asymptotic (i.e. assume infinitely large sample sizes); or they require very restrictive and/or unrealistic

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[15]Relying on the central limit theorem (CLT) in this way requires additional assumptions, such as the finiteness of the mean and the variance of the distribution of the data, and under some conditions that the data is independent and identically distributed ("iid"). Also, the size of the sample required for the convergence of the sample mean's distribution to the Gaussian distribution can rely heavily on the distribution of the underlying data.

assumptions about the distributions of the returns data (e.g. that they are multivariate Gaussian, or elliptical; or even that they are independent and identically distributed ("iid"), or all symmetric, or all stationary, etc.); or they require very restrictive and/or unrealistic assumptions about the values of the dependence measures themselves (e.g. the cells are all zeros, or all equal to each other, or follow very discrete and limited block structures); or they estimate the all-pairwise matrix in ways that do not guarantee its positive definiteness; or, most typically, they require multiple of these unrealistic assumptions and restrictions combined.

But in real-world practice, what we need for useful and useable application to answer the kinds of inferential questions listed above is the *finite-sample* **distribution** of ALL of the relevant dependence measures (for ceteris paribus comparisons) *under real-world data conditions*, that is, non-iid multivariate financial returns data with marginal distributions that have different and varying degrees of asymmetry, serial correlation, (non)stationarity, and heavy-tailedness. The finite sample distribution of the matrix also must be valid for all potential values of the dependence matrix, regardless of the estimator being used. The inferential results that depend on this distribution, i.e. the p-values and confidence intervals, must be provided at both the matrix level and the individual pairwise cell levels, and these results must be analytically consistent across the levels. And the method generating the distribution that provides all of these results must remain valid even when the matrix is restricted by the dictates of a particular Scenario, for which many cells remain 'frozen' (i.e. unaffected by the scenario); for example, many pairwise associations strongly affected in a Covid-type scenario will remain unaffected under a housing bubble scenario.

The method derived in the coming chapters – Nonparametric Angles-based Correlation, or "NAbC" – checks all of these boxes, just as described in the Introduction. This allows for valid statistical inference, under real-world conditions, without unrealistic and restrictive assumptions on the data or the values of the all-pairwise matrix. NAbC not only provides answers to any of the inferential questions posed above, and then some, but also does so using the one single method for defining the finite-sample distribution of the dependence matrix. This allows for ceteris paribus comparisons of different dependence measures, and different estimators of those dependence measures, under controlled conditions so that we can choose the best among them (i.e. which is more robust, powerful/precise, accurate, with shortest average run-length, etc., all else equal) depending on the data conditions and/or the hypotheses being investigated. The alternative is to use an incomplete patchwork of disparate derivations of distributions, some asymptotic, some not, valid under questionably similar but usually very restrictive and typically unrealistic conditions that, in the end, prohibit reliable comparative analyses and are more confounding than elucidating.

Before deriving and implementing NAbC, and demonstrating its broad scope and flexibility under real-world data conditions, I first complete this section with additional, important dependence measures that all fall within NAbC's range of application, and for the newer ones, are increasingly used in practice.

<u>Tail Dependence</u>

Another important and time-tested dependence measure, especially for risk analyses, is the tail dependence matrix (TDM). Conceptually, TDM measures the probability of a variable (return) value residing in the tail of one variable's distribution given that the value of the other asset resides in the tail of it's distribution. More precisely, TDM provides the probability of a variable exceeding a quantile of its distribution conditional on the other variable in the pair exceeding the same quantile of its distribution. Hence, the tail dependence matrix consists of conditional probabilities of quantile exceedance, so each value can range from zero to one, rather than -1 to 1 like the "big 3." But otherwise the matrix conditions listed in (4) above all hold (its positive definiteness is proven later herein, and was proven by Embrechts et al., 2016). The upper tail dependence matrix only is equal to the lower tail dependence matrix if data distributions are perfectly symmetric: otherwise, the two metrics have distinct values, as shown below in (5) and (6):

(5) $TDMU_{X,Y} = \lim_{q \to 1^-} P\left(Y > F_Y^{-1}(q) \mid X > F_X^{-1}(q)\right)$

(6) $TDML_{X,Y} = \lim_{q \to 0^+} P\left(Y \le F_Y^{-1}(q) \mid X \le F_X^{-1}(q)\right)$

where quantile function = inverse cdf = $F^{-1}(q) = \inf\left\{x \in \mathbb{R} : F(x) \ge q\right\}$

Other measures of tail dependence exist (see AghaKouchak et al., 2013, Babić et al., 2023, Manistre, 2008, Li and Joe, 2024, Krupskii and Joe, 2014, Lauria et al., 2021, and intriguingly, Siburg et al., 2024), but (5) and (6) are the oldest, most widely used, and best understood. Tail dependence is especially important in the risk analytics of financial portfolios because "tail events" often represent the most material financial impacts, are typically associated with non-linear effects and associations, and are closely tied to correlation breakdowns: as is commonly and rightly stated, "when things go bad they go bad together." The eponymous phenomenon of correlation breakdowns is treated in more detail later in this monograph, but note that the tail dependence matrix has been one of the principal tools used in both the literature and by practitioners to quantitatively estimate, model, and mitigate it.

<u>More Recent Dependence Measures</u>

The design of a more recent and flexible dependence measure, Szekely's distance correlation (Szekely et al., 2007), seeks to better handle dependence that is both non-linear and non-monotonic. It uses two matrices: the matrix of pairwise distances between all values in a sample from X, and the same matrix calculated from a sample from Y. To the extent that these matrices vary together, Szekely's distance correlation will approach a value of 1, and to the extent they do not, it will approach a value of zero. So its range is zero to one and a value of zero, unlike the "big 3," DOES indicate independence between X and Y. Also unlike the "big 3" its value does not indicate with a positive or negative sign whether dependence between X and Y is positive or negative. Notably, the distance correlation can be calculated in arbitrary –

and different – dimensions, so the sample from X can be drawn, for example, from a three dimensional distribution, and the sample from Y can be drawn from a six dimensional distribution.

(7) first, create n x n distance matrices a and b by letting

$$a_{i,j} = \left\| x_i - x_j \right\| \text{ and } b_{i,j} = \left\| y_i - y_j \right\|, \ i,j = 1,2,3,...,n \text{ where } \left\| \text{vector } z_n \right\| = \sqrt{z_1^2 + z_2^2 + \cdots + z_n^2}$$

Next, subtract from a and b their row and column means, and add their respective matrix means, as shown below:

$$A_{i,j} = a_{i,j} - a_{*,j} - a_{i,*} + a_{*,*} \text{ and } B_{i,j} = b_{i,j} - b_{*,j} - b_{i,*} + b_{*,*}$$

Then Szekly's distance correlation = $dcorr = \sqrt{ \dfrac{1}{n^2} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} A_{i,j} B_{i,j} } \Bigg/ \sqrt{ \dfrac{1}{n^2} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} A_{i,j}^2 \cdot \dfrac{1}{n^2} \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} B_{i,j}^2 }$

Another recent dependence measure – Lancaster's correlation (see Holzmann and Klar, 2024) – shares several characteristics with Szekely's: its values range from zero to one, a value of zero indicates independence, and it does not indicate with a positive or negative sign whether the dependence between X and Y is positive or negative. Lancaster's correlation was designed to not only handle non-linear and non-monotonic dependence, but also to improve upon, via increased robustness and generalizability and ease of computation, another dependence measure, the maximal correlation (see Hirschfeld (1935) and Gebelein (1941)).

(8) $lan = \max \left(\left| r\left(\tilde{X}, \tilde{Y} \right) \right|, \left| r\left(\tilde{X}^2, \tilde{Y}^2 \right) \right| \right)$ where $\tilde{X} = \Phi^{-1}\left(F_X(X) \right)$ and $\tilde{Y} = \Phi^{-1}\left(F_Y(Y) \right)$, where r is Pearson's

correlation, $\vert \ \vert$ is the absolute value function, Φ^{-1} is the quantile (inverse cdf) function of the standard normal distribution, and F is the (empirical) cdf of each variable.

A second version is called linear Lancaster's correlation:

(9)

$$lanL = \max \left(\left| r(X,Y) \right|, \left| r\left(\breve{X}^2, \breve{Y}^2 \right) \right| \right) \text{ where } \breve{X} = \left(X - \bar{X} \right) \Bigg/ \sqrt{ \sum\limits_{i=1}^{n} \left(X_i - \bar{X} \right)^2 \Big/ (n-1) } \text{ and } \breve{Y} = \left(Y - \bar{Y} \right) \Bigg/ \sqrt{ \sum\limits_{i=1}^{n} \left(Y_i - \bar{Y} \right)^2 \Big/ (n-1) }$$

$$\text{and } \bar{X} = \sum\limits_{i=1}^{n} X_i \text{ and } \bar{Y} = \sum\limits_{i=1}^{n} Y_i$$

Holzmann and Klar (2024) conduct empirical analyses comparing Szekely's distance correlation and both Lancaster's correlations under a wide range of data conditions. They also compare these to another new, but asymmetric dependence measure, called Chatterjee's correlation coefficient.

Asymmetric, Directional Dependence Measures

Chatterjee's correlation coefficient garnered much attention upon its publication in 2021. This is largely due to its simplicity and ease of implementation as a measure of non-linear, non-monotonic, regression-based, and cyclical dependence. If X and Y pairs are ranked according to X values, with no ties on the X values, so that $\left(\left(X_{(1)}, Y_{(1)} \right), \left(X_{(2)}, Y_{(2)} \right), \cdots, \left(X_{(n)}, Y_{(n)} \right) \right)$ then:

$$(10) \quad chcorr = \xi_n\left(X, Y\right) := 1 - \frac{3\sum_{i=1}^{n-1} \left| r_{i+1} - r_i \right|}{n^2 - 1} \quad \text{where } r_i = \text{rank of } Y_i$$

Under ties for some of the X values, break ties uniformly at random, and

$$(11) \quad chcorr = \xi_n\left(X, Y\right) := 1 - \frac{n\sum_{i=1}^{n-1} \left| r_{i+1} - r_i \right|}{2\sum_{i=1}^{n} l_i \left(n - l_i \right)} \quad \text{where } l_i = \#j \text{ such that } Y_{(j)} \geq Y_{(i)}$$

Unlike the big 3, Chatterjee's new correlation coefficient ranges from zero to one asymptotically (it can exceed these bounds slightly under finite samples), and a value of zero does indicate independence. Also, no positive or negative dependence is indicated by a positive or negative sign on the measure value. Most notably, this is an asymmetric dependence measure, that is, the order of X and Y matters: $\xi_n\left(X, Y\right)$ does not necessarily equal $\xi_n\left(Y, X\right)$. In other words, the dependence of Y on X is not assumed to be identical to the dependence of X on Y, respectively.[16] However, note that Chatterjee's can be made to be symmetric by simply taking the maximum of two measures, one in each direction as in (12):

$$(12) \quad chcorr\_sym = \max\left[\xi_n\left(X, Y\right), \xi_n\left(Y, X\right) \right]$$

Chatterjee's breakthrough has spawned many variants (see Lin & Han, 2023, Pascual-Marqui et al., 2024, and especially Gao and Li, 2024). One of these is the "improved Chatterjee's correlation" derived by Xia et a. (2024), the motivation of which is to increase power by using inverse distance weightings of all neighboring data values as opposed to just one.

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[16] It is important to note that herein, when using dependence measures that are asymmetric/directional, the corresponding all-pairwise matrix remains symmetric. So when using, say, Chatterjee (2021), on the returns of two particular assets in the portfolio, say, X3 and X4, the value in cell row 3, column 4 of the matrix is $\xi_n\left(X3, X4\right)$, and the value in cell row 4, column 3 of the matrix is identical, that is, $\xi_n\left(X3, X4\right)$; it is NOT $\xi_n\left(X4, X3\right)$.

$$(13) \quad ichcorr = \xi_n^{IM}(X,Y) = 1 - \frac{\sum\limits_{i \neq j}^{n} |r_i - r_j| / |i - j|}{\frac{n+1}{3} \sum\limits_{i \neq j}^{n} |i - j|}$$

Xia et a. (2024) test the power and level of "improved Chatterjee" against both Chatterjee and Szekely's correlations in an empirical study under widely varying data conditions. Both Chatterjee's and improved Chatterjee's coefficients exhibit power under non-monotonic, non-linear, and cyclical dependence, with the latter usually winning.[17]

Interestingly, Zhang (2023) has proposed combining Chatterjee's and Spearman's in an effort to obtain the best of both worlds: a dependence measure that has reasonable power under cases of non-monotonic, non-linear, and/or cyclical dependence (where Spearman's has little to no power, especially compared to Chatterjee's) as well as reasonable power under monotonic dependence (where Chatterjee's has less power than Spearman's).

$$(14) \quad zcorr = I_n(X,Y) = \max\left\{ |sr_{X,Y}|, \sqrt{5/2}\,\xi_n(X,Y) \right\}$$

Zhang's (2023) combined correlation ranges from 0 to 1, where zero indicates independence. This dependence measure also is asymmetric due to its inclusion of Chatterjee's coefficient.

Finally, asymmetric, directional dependence measures also can be applied only to the tails of X and Y, and it is important to note that correlation breakdowns often are associated specifically with (asymmetric) tail dependence: "Extensive evidence has been gathered showcasing the prevalence of heavy-tailed distributions and asymmetric tail interdependence within equity and foreign exchange markets, particularly during times of crisis. …This phenomenon causes markets that typically exhibit minimal or no correlation to behave similarly, often in opposition to fundamental principles." (Pramanik, 2024). One straightforward example of an asymmetric tail dependence measure is that of Deidda et al, (2023) which is essentially Kendall's Tau applied conditionally, only when the percentile, q, of X (or Y) is exceeded:

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[17] Note that both Chatterjee (2021) and Xia et a. (2024) test against two dependence measures not explored further herein: the HSIC measure of Gretton et al. (2007), and the HHG measure of Heller et al. (2013). Both appear to have excellent power under circular and heteroskedastic data, and the former maintains reasonably large power under other conditions where both Chatterjee statistics outperform it. While both HSIC and HHG are much more computationally intensive than either Chatterjee dependence measure, Sejdinovic et al. (2013) intriguingly prove that "reproducing kernel Hilbert space (RKHS)-based dependence measures are precisely the formal extensions of the [Szekely et al. (2007)] distance covariance." So HSIC is a generalized version of Szekely et al. (2007) that circumvents "the problem of nonintegrability of weight functions by using translation-invariant kernels called distance-induced kernels." RKHS-based dependence measures remain an active and intriguing area of continuing research (for examples, see Ke, 2019; Mitchell et al., 2022; Tripathi et al., 2022; Wahba, 2017; and Zhang & Songshan, 2023).

$$(15)\ \hat{\tau}_{X,Y}(q) = \frac{1}{\binom{k}{2}} \sum_{1 \le i \le j \le n} \text{sgn}(X_i - X_j)\,\text{sgn}(Y_i - Y_j)\,I\left(X_i, X_j > X_{(n-k)}\right)$$

where $q = 1 - k/n$, and $k \le n$ is the number of exceedences used in the tail, and $I(\ )$ is the indicator function (one when true, zero otherwise) ensuring that only the k largest observations of X are used. Note again that generally, $\hat{\tau}_{X,Y}(q) \ne \hat{\tau}_{Y,X}(q)$, that is, this tail dependence measure is directional, and the affect of X's tail on Y's tail is not assumed to be the same as that of Y's tail on X's tail.

Other directional, asymmetric dependence measures include the QAD measure of Junker et al. (2021), and others described in Jondeau (2016).[18]

It remains notable that NAbC's broad scope allows for its application to these asymmetric, directional dependence measures as readily as it is applied to the big 3. As seen in a later chapter, this gives NAbC great utility in some surprising settings. Even as it is designed fundamentally as a method for robust statistical inference, when using these directional dependence measures it appears NAbC can be applied to increase the power of causal models to accurately recover directed acyclic graphs (DAGs). This is an area of continuing research, but serves as an example of how NAbC's breadth of application can be useful even beyond ceteris paribus comparisons of the inferential power of competing dependence measures. Yet this remains invaluable as it stands, as such comparisons often would not be possible without NAbC. All dependence measures have strengths and weaknesses not only under data different conditions, but also depending on the specific questions applied researchers and practitioners need to answer, so we need to be able to test them, using the same unifying method under controlled conditions, to determine which is most appropriate for a given situation.

However, the main characteristic that unifies them for purposes of this monograph is that the all-pairwise matrix of all of them is positive definite. This is proven mathematically below, and covers essentially all dependence measures in widespread usage, both in finance and other applied, empirical settings. So requiring positive definiteness really is no restriction at all, and allows for NAbC's very widespread application in practice. It's ability to define all these dependence measures' finite-sample distributions allows it to be the unifying method of choice, if not the only method in many cases, for answering the types of inferential questions posed above, especially for comparative, ceteris paribus analyses.

To conclude this chapter, it is important to reemphasize here that several of the dependence measures listed above (e.g. Szekely's, as well as some variants of Chatterjee's (see Pascual-Marqui et al., 2024)), can be applied on a multivariate basis, in arbitrary dimensions (see Chatterjee (2024) and Han (2021) for useful surveys of these, in addition to Grothe et al. (2014), Latif and Morettin (2014), Reddi et al. (2015), Li

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[18] Note that under certain conditions, such as when categorical and ordinal data are being analyzed and the number of categories between the two variables differs dramatically, even Pearson's correlation can be unambiguously directional (see Metsämuuronen, 2022, for details).

and Joe (2024), Yu et al. (2021) and Puccetti (2022) for some approaches not covered herein[19]). But "positive definite" is not applied in this multivariate sense (see Cardin, 2009); herein it is applied strictly to the all-pairwise matrix, which contains only bivariate relationships. I explain further below in the Scenarios chapter some of the reasons for using the all-pairwise framework here, which arguably is most commonly used in real-world financial settings, as it remains highly flexible, typically provides more statistical power compared to multivariate approaches, and simultaneously allows for more precise attribution and intervention 'what if' analyses.

With this brief but important review of relevant and widely used dependence measures aside, I address their estimation in the next chapter, before turning to the derivation of NAbC in subsequent chapters.

Estimation

Regarding estimation of the all-pairwise matrix, the literature focuses almost exclusively on estimators for the covariance matrix and the Pearson's correlation matrix. This is not terribly surprising given the relatively long history and widespread usage of Markowitz's portfolio framework (see Markowitz, 1952) and related models.

"Accurate covariance matrix prediction is crucial for portfolio optimization and risk management because it captures the relationships and co-movements between asset returns." (Lee et al., 2024)

But fortunately, some see the bigger picture, that these analyses can and should be broadened to ALL positive definite dependence measures

"Modeling covariance matrices – or more broadly, positive definite (PD) matrices – is one of the most fundamental problems in statistics" (Lan et al., 2020).

So I first focus below on estimation of Pearson's matrix, and later on the other dependence measures discussed in the previous chapter.

The first of the two major challenges of estimating the all-pairwise matrix of any dependence measure is sample size, because we are not just estimating a single parameter, say, a volatility or a beta of a single asset, but rather , $\binom{p}{2} = \dfrac{p(p-1)}{2}$ pairwise associations. To do this accurately and with reasonable

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[19] Note that the primary focus of the development of many multivariate dependence measures is on testing the null hypothesis of (multivariate) independence, and thus, on the level and power of this specific test for these measures. While this objective is foundational, that of this monograph is not only on this binary question, but also on the practical and accurate measurement of the magnitude of such dependence (measured using bivariate associations) when it exists. Consequently, I focus on dependence measures in the literature with strong results related to their statistical power, level control, ease of implementation, low computational complexity, and attainment of the full range of values they are meant to attain under the relevant sample spaces. More importantly, relying on the bivariate, as opposed to multivariate, relationships measured in the all-pairwise matrix is critical to the scenario flexibility provided by NAbC, as explained in later chapters.

precision, we need more data than is needed for a single estimate. Regarding accuracy, the sample covariance matrix, and thus, the sample Pearson's matrix are consistent estimators, that is, they are asymptotically unbiased. But regarding precision, their estimates will be way too variable to be useable or useful, not to mention biased in the finite-sample case, in the absence of large(r) data samples. A widely recognized rule of thumb is that the sample size needs to be at least ten times the dimension of the matrix (N≥10p; see Bongiorno et al., 2023), but this arguably depends on the method used to estimate it. For example, Bun et al. (2016) devise a rotationally invariant estimator that "cleans" or "de-noises" the estimate of Pearson's matrix using functions of its eigen values, a method for which they argue that N≥2p is sufficient. Note that the estimators of all the dependence measures presented herein are at least asymptotically unbiased, and that some researchers believe the sample size issue has been addressed as well as it can be, especially if the best methods are being used (see Bouchaud, 2021: "Now the data problem is solved as best as possible…" referring to Bun et al., 2023, among others). With this in mind, currently it would appear that Bun et al. (2016) is the state-of-the-art estimator for the **unconditional** estimate of Pearson's matrix (see du Plessis & van Rensburg (2020) for a comparison study). But unconditional estimates assume the values of the matrix do not change over time, that is, that the financial time series data is stationary. And this brings us to the second major challenge when estimating any all-pairwise dependence matrix: non-stationarity.

As Bouchard (2021) rightly points out, portfolio frameworks like that of Markowitz (1952), and really any in applied usage, require knowledge of the dependence measure (here, Pearson's) to be representative of the future realized correlations, because financial data is not stationary (i.e. its conditional distribution changes over time). Therefore, we need a forecast, into the near-term future, of the **conditional** Pearson's matrix. And a very compelling one, namely, "Average Oracle" (AO), is exactly what is provided by Bongiorno et al. (2023) (see also Bongiorno & Challet, 2023a, for an extensive empirical study against competitors). Conceptually AO is very straightforward: based on the eigen decomposition of Pearson's matrix, it uses the (oracle) covariance of the next-period 'future' with the eigenvectors of the adjacent past period to obtain eigenvalues that, when averaged over many samples, embed the desired, dynamic time effects for a robust forecast of Pearson's matrix. Somewhat surprisingly, this intuitive method outperforms all flavors of advanced "shrinkage," both non-linear (see Ledoit & Wolf, 2017) and quadratic (see Ledoit & Wolf, 2022a, 2022b) as well as DCC and NLS combinations (see Engle et al., 2019). It is fast, straightforward to understand and implement, and importantly, fully nonparametric. AO's outperformance of the widely used NLS approach perhaps should not be so surprising given that Bongiorno & Challet (2023b) recently proved that NLS is not optimal for portfolio optimization, as was widely believed, because it does not optimize under non-stationarity.[20] So I recommend AO as the current state-of-the-art conditional estimator of Pearson's matrix. However, this literature is vast, comprising easily many hundreds of papers if covariance estimation is included, and given the current rapid pace of research in this area, it is certainly possible that new, worthy competitors exist, especially

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[20] For those that view shrinkage favorably in general, an improved shrinkage competitor with arguably better properties than NLS is that of Kelly et al. (2024)).

under specific data conditions (see include Zhang et al. (2022), Zhang et al. (2023), Vanni et al. (2024) and Zhangshuang et al. (2025) for such recent possible examples).

For estimation, conditional or unconditional, of the all-pairwise matrices based on the other dependence measures listed in the previous section, the literature has little to offer beyond the fact that all of the sample estimators presented in the previous chapter are at least asymptotically unbiased (see Zhao et al. (2014) for an exception). So as long as sample sizes are sufficiently large these estimators will retain good statistical properties. However, I offer two additional suggestions below for possible improvements, the latter of which is an active research paper I am pursuing. The first is simply the inverse of a common robustification technique using a well established relationship between Pearson's and the rank-based measures, Kendall's and Spearman's. Estimates of (bivariate) Kendall's Tau or Spearman's Rho often are used to robustify those of Pearson's using the widely known relationships of $r = \sin(\tau \pi/2)$ and $r = 2\sin((sr)\pi/6)$, respectively, which are valid under iid elliptical data distributions (see Sheppard, 1899; Greiner, R., 1909; Lindskog et al., 2003; Heinen & Valdesogo, 2022; McNiel et al., 2005;and Hansen & Luo, 2024; and for advanced methods on this, see Barber & Kolar, 2018, and Niu et al., 2020). Yet under specific, known data conditions that are non-elliptical, it may be demonstrable that these transformations remain reasonably accurate (see Hamed (2011) and Hansen & Luo (2024) for examples). In this case, given a strong estimator of Pearson's from an improved estimation method like those described above (e.g. Average Oracle of Bongiorno et al. (2023)), the inverses of these functions could be used to obtain estimates for the all-pairwise matrices of Kendall's and Spearman's that likely would share some of the benefits of an improved estimate for Pearson's matrix, especially when it is conditional. Note, however, that these transformations would require verifying, and sometimes enforcing, positive definiteness ex post (see McNeil et al. (2015) and Higham (2002)). Also, they are limited to Kendall's and Spearman's.

However, a much broader and more promising (if still unproven) approach, would be to use Average Oracle on ANY of the above-mentioned measures directly, simply replacing Pearson's matrix but keeping the methodology otherwise identical. Again, because their all-pairwise matrices all are positive definite, they will have valid eigen decompositions wherever the covariance matrix will (and even in some cases where the covariance matrix is singular), and the 'training' eigenvectors of the adjacent past can be used in exactly the same manner with next-period 'future' all-pairwise dependence matrices to obtain (averaged) eigenvalues that embed the measured, average, empirical time dependency. This is the subject of my continuing research, but like the original method of Bongiorno et al. (2023), preliminary results appear very promising.

Promising additional approaches to estimation aside, all the above reemphasizes the fact that NAbC is not an estimator of any of these dependence measures: rather, it provides the finite-sample distribution of the estimate, for any estimator of any dependence measure (that is positive definite) under any real-world data conditions. This allows us to make actionable inferences about dependence structure in a unified way, allowing for comparative, ceteris paribus analyses. The literature to date provides such distributions in a highly piecemeal fashion for *some* of the dependence measures under *some* (often very

limited and/or unrealistic) data conditions for *some* (often very limited and/or unrealistic) ranges of values. The derivations often are extremely complex and unwieldy and unusable for many practitioners. NAbC sidesteps all of those problems with a single, unified, and straightforward method. Estimation of the all-pairwise matrix is the only thing out-of-scope for NAbC, but in a sense this is a strength of the approach since it permits NAbC to remain "estimator agnostic," allowing for its application on any reasonable and relevant estimator of the all-pairwise matrix and providing the flexibility to use those that are most robust and/or most precise and/or most accurate – or any combination thereof – under different conditions. So I do not need to reinvent the already well-made wheel of estimation here[21]: we get tremendous benefit, not previously attainable, by applying NAbC to obtain the *finite sample distribution* of the estimate, and thus, make statistically valid and actionable inferences about our portfolio's dependence structure. Derivation and application of NAbC follows below in the next chapter.

NAbC: (Robust) Statistical Inference

Brief Literature review of Pearson's Matrix: Distributional Results and Sampling Algorithms

I begin with Pearson's product moment correlation matrix, the oldest and arguably most widely used measure of dependence. Although its limitations often are mischaracterized or misunderstood, especially as they relate to widely held views classifying it strictly as a measure of linear association (see van den Heuvel & Zhan, 2022), in many settings it remains either optimal or centrally relevant for wide-ranging purposes.[22] These include robust asset allocation (Welsch and Zhou, 2007), Black-Litterman variants (Meucci, 2010a, Qian and Gorman, 2001), entropy pooling with fully flexible views (Meucci, 2010b), portfolio optimizations combined with random matrix theory (Pafka and Kondor, 2004), stress testing (Bank for International Settlements, Basel Committee on Banking Supervision, 2011a), and even non-linear, tail-risk-aware trading algorithms (Li et al., 2022, and Thakkar et al., 2021) to name a few. Consequently, Pearson's is the foundational dependence measure we start with (see also Rodgers & Nicewander (1988) for a broad, useful, and applied introduction to Pearson's).

When it comes to statistical inference and simulation-based decision-making, the extant literature on Pearson's matrix can be placed roughly into two categories: 1. distributional derivations that preserve inferential capabilities, but usually under narrow, limiting, and often unrealistic constraints; and 2. sample-generating algorithms that either share the same limitations, or attempt to generate stylized, real-world distributions but fail to preserve inferential (probabilistic) validity. The latter often are used indirectly for such purposes as scenario analytics and stress testing. Of course, we want both worlds:

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[21] With the possible exception, which I currently am researching, of applying AO to dependence measures beyond Pearson's, which could be a notable improvement, as it is for Pearson's, over any other method for forecasting their conditional values.
[22] In addition to the linearity 'myth' effectively addressed in van den Heuvel & Zhan (2022), note also that while Pearson's, under dependence, does not retain invariance under marginal transforms generally, the set of cases where it *does* retain invariance is broader than previously thought (see Koike et al., 2024).

robust, fast, straightforward algorithms to generate samples when needed (i.e. in the absence of fully analytic solutions), that also preserve inferential capabilities, so that we can base consequential decisions on rigorously defined probabilities.

I begin with a brief and admittedly non-comprehensive, but well-targeted literature review of 1. and then 2. under typically more restricted cases, and then treat both under more general conditions. Subsequently I develop NAbC under both a narrowly defined but foundational case, and then fully general conditions that satisfy the eight original objectives. Defining NAbC under a narrow case provides a fully analytic version of it that very transparently shows how NAbC accomplishes both purposes above – useful sample generation and valid statistical inference, simultaneously – while also serving as a helpful referential baseline for NAbC's generalization to all dependence measures, under all data conditions.

Distributional Results

Derivations of the distribution of Pearson's matrix go all the way back to the father of modern statistics, Sir Ronald A. Fisher (see Fisher, 1915, 1928). Intriguingly, Fisher (1928) recognizes the relationship between the Pearson's correlation formulae and the cosine between the angles of the two data vectors in the bi-variate case, i.e. what is now widely referred to as "cosine similarity" (described in more detail later).[23] He builds on this in his derivations (as does NAbC below), and although without closed forms some of the mathematical results prove unwieldy, they are foundational for those (re)derived below. Joarder and Ali (1992) replicate some of Fisher's earlier results (see Fisher, 1915), and more generally derive the distribution of Pearson's for any dimension when the underlying data is elliptically distributed (which includes the case of Gaussian data). Their density, however, requires iterated integration on the order of the dimension of the matrix, so like many of Fisher's results, while mathematically correct, it remains unscalable and less readily implemented.

For more recent results, below I start with narrowly defined cases and then expand. Restrictions on the narrow cases include i. on the underlying data (e.g. only Gaussian); ii. on the dimension of the matrix (e.g. only the bivariate case of p=2); iii. on the values of the matrix (e.g. only the identity matrix, where all correlations equal zero); and iv. with a priori known, rather than estimated, parameter values (e.g. known variances).

- <u>Gaussian data, any matrix, p=2</u>

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[23] Briefly, this widely used mathematical relationship recognizes that the cosine of the angle between two mean-centered data vectors equals Pearson's (bivariate) correlation coefficient, as below:

$$(16) \quad \cos\left(\hat{\theta}\right) = \frac{\text{inner product}}{\text{product of norms}} = \frac{\langle \mathbf{X}, \mathbf{Y} \rangle}{\|\mathbf{X}\|\|\mathbf{Y}\|} = \frac{\sum_{i=1}^{N}\left(X_i - \sum_{j=1}^{N}X_j\right)\left(Y_i - \sum_{j=1}^{N}Y_j\right)}{\sqrt{\sum_{i=1}^{N}\left(X_i - \sum_{j=1}^{N}X_j\right)^2}\sqrt{\sum_{i=1}^{N}\left(X_i - \sum_{j=1}^{N}Y_j\right)^2}} = \frac{Cov_{X,Y}}{s_X s_Y} = r_{X,Y}, \ 0 \leq \hat{\theta} \leq \pi$$

For Gaussian data with matrix dimension p=2, i.e. the bi-variate case, Taraldsen (2021) derived the exact confidence distribution of Pearson's correlation:

$$(17)\ \pi(\rho\,|\,r) = \frac{\left(1-r^2\right)^{(v-1)/2} \cdot \left(1-\rho^2\right)^{(v-2)/2} \cdot \left(1-r\rho\right)^{(1-2v)/2}}{\sqrt{2}\,B(v+1/2,\,1/2)} \cdot {}_2F_1\left(\frac{3}{2},-\frac{1}{2};v+\frac{1}{2};\frac{1+r\rho}{2}\right)\ \text{where}$$

$B(X,Y) = \left[\Gamma(X)\Gamma(Y)\right]/\Gamma(X+Y)$ the Beta function, $v = n-1 > 1$, and F is the Gaussian hypergeometric function where

$${}_2F_1[a,b;c;z] = \sum_{n}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \cdot \frac{z^n}{n!}\ \text{where}\ (h)_n = h(h+1)(h+2)\cdots(h+n-1),\ n\geq 1,\ (h)_0 = 1\ ^{24}$$

It is important to note here that Taraldsen (2021) shows that the Fisher's Z-transformation (Fisher, 1921), a widely used approximation of this distribution, loses accuracy as correlation values approach one or negative one, especially for smaller samples.

- Gaussian data, identity matrix only, p≥2

For the Gaussian identity matrix (all correlations of zero) with p≥2, Gupta & Nagar (2000) derive the density

$$(18)\ f(R) = \frac{\left[\Gamma\left([n-1]/2\right)\right]^p |R|^{(n-p-2)/2}}{\Gamma_p\left([n-1]/2\right)},\ -1\leq r_{i,j} = r_{j,i} \leq 1,\ r_{i,i} = 1,\ 1\leq i,j\leq p\ \text{and}\ \Gamma_p\left([n-1]/2\right) = \pi^{[p(p-1)/4]}\prod_{i=1}^{p}\Gamma\left([n-i]/2\right)$$

Although Pham-Gia & Choulakian (2014) claim this is a new result, it actually is identical to that of Joarder and Ali (1992) under these conditions (see (4.2)), and after some manipulation, that of Fisher (1915) for the bivariate case (see (4.2), (4.3), and (3.1) in Joarder & Ali, 1992).

- Gaussian data, any matrix, with p≥2:

Under Gaussian data, with p≥2, Pham-Gia & Choulakian (2014) provide the distribution of the sample Pearson's matrix under any values, not just the identity matrix:

$$(19)\ f(R) = \frac{\left[\Gamma\left([n-1]/2\right)\right]^2 \exp\left\{-\sum_{i<j}\frac{\lambda_{i,j}s_{i,j}}{\sqrt{\sigma_{i,i}\sigma_{j,j}}}\right\}}{\Gamma_p\left([n-1]/2\right)\left[|\Lambda|\prod_{i=1}^{p}\lambda_{i,i}\right]^{([n-1]/p)}} |R|^{(n-p-2)/2}$$

with sample covariance $\{s_{i,j}\}$, $1 \le i < j \le p$, $\Gamma_p\left(\left[n-1\right]/2\right) = \pi^{\left[p(p-1)/4\right]} \prod_{i=1}^{p} \Gamma\left(\left[n-i\right]/2\right)$,

known variance $\{\sigma_{i,i}\}$, $|\ |$ is the determinant function, Λ is the true correlation matrix, and $\lambda_{i,i}$ the diagonals of Λ^{-1}

The limitations of Pham-Gia & Choulakian (2014) include the requirement of a priori knowledge of true (not estimated) variances, and of course, its validity only for normally distributed data. It also arguably is quite cumbersome to implement.

Sampling Algorithms

Moving now to sample generation under various 'non-generalized' conditions, i.e. conditions that are not generalized to those common in financial portfolios, the literature provides a number of methods, many of which are quite involved. Note that I have focused on more recent ones, as these usually explicitly subsume previously published algorithms, and many of the below are even compared against each other in their own empirical simulation studies. Note that none of these are tailored to generate the stylized empirical characteristics observed in financial portfolios, and hence are labelled here as 'non-generalized'.

i. The onion and c-vine methods, the former of which can generate random correlation matrices with the joint density of the correlations being proportional to a power of the determinant of the correlation matrix, and the latter of which is based on partial correlations specified in a vine copula (specifically, a c-vine copula). (Lewandowski et al., 2009)
ii. the chordal sparsity method of Kurowicka (2014), which generalizes Lewandowski et al. (2009), although "it is not clear whether it is possible to extend them to other patterns of unspecified correlations" beyond those with chordal sparsity patterns.
iii. The restricted Wishart distribution approach of Wang et al. (2018), which is equivalent to Lewandowski et al. (2009) but somewhat more efficient.
iv. The hyperspherical coordinate approach of Pourahmadi et al. (2015)
v. The Cholesky-Metropolis method of Cordoba et al. (2018), which claims to be faster than the previously listed methods.
vi. The direct formulation method of Madar (2015)
vii. The flexible bijection method of Veleva (2017)
viii. The rejection algorithm of Makalic and Schmidt (2018), which is based on the polar (hyperspherical) angles representation of Pearson's matrix[25]

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[25] See Joarder & Ali (1992), Pinheiro & Bates (1996), Rebonato & Jaeckel (2000), Rapisarda et al. (2007), and Pourahmadi & Wang (2015). The use of spherical angles for analysis of Pearson's matrix goes back at least to Fisher (1915), but Joarder & Ali (1992) and Rapisarda et al. (2007) provide a geometrically motivated, thorough, and clear descriptions of its derivations, and Rebonato & Jaeckel (2000) appears to have been the first to propose its application in financial settings.

Makalic and Schmidt (2018) is treated in more detail below. Implementation of all but vi., vii., and viii. above arguably remain quite involved, but one of the self-described focuses of most of these is computational efficiency (which is not surprising as they are sampling algorithms).

"This new method is faster than the original onion method for generating random matrices, especially in the low dimension (T < 120) situation." Wang et al., 2018

"...comparing the HP and C-Vine algorithms, it is evident that the HP algorithm is faster for larger n and is slower for smaller n." Pourahmadi et al., 2015

"We have also executed a comparative performance study, where our approach has yielded faster results than all of the related approaches in the literature." (Cordoba et al., 2018), comparing against the c-vine and onion methods of Lewandowski et al. (2009) and the hyperspherical approach of Pourahmadi et al. (2015), but not Makalic and Schmidt, 2018

"...emphasizes the efficiency of the proposed algorithm." (Makalic and Schmidt, 2018)

From a close read of the runtime results of the successively published and compared algorithms above, it appears that Makalic and Schmidt (2018) is the fastest among them (excepting those of Madar (2015) and Veleva (2017), which have not been compared to the others). However, as discussed in more detail below, Roman (2023) shows that for the case of the Gaussian identity matrix, when NAbC is used as a sampler, it is over 30% faster than Makalic and Schmidt (2018); when NAbC is used analytically, its results are, for all intents and purposes, instantaneous, as can be seen in the excel workbook at the following link (see http://www.datamineit.com/JD%20Opdyke--The%20Correlation%20Matrix-Analytically%20Derived%20Inference%20Under%20the%20Gaussian%20Identity%20Matrix--02-18-24.xlsx). Runtimes of NAbC under the fully general case, i.e. that of fully realistic financial data conditions (which none of the algorithms above claim to cover), as well as any valid matrix values, are discussed in later chapters below.

But beyond speed, the more important issue regarding these sampling algorithms is that none preserve inferential validity, on a sample-by-sample basis, by providing a readily calculated cumulative distribution function (cdf) value (probability density function (pdf) values will be more cumbersome and less useable here). In other words, to make these simulation results truly useful for precise, powerful hypothesis testing and other inferential purposes, we need to know where in the distribution of correlation matrix samples a particular correlation matrix sits: what probability is it associated with? Calculating this (cdf value) is tortured, if not impossible for these methods, although this arguably should be the primary focus of such algorithms.

The counter argument is that the group of samples that these algorithms generate, taken as a whole, is a valid representation of the data generating mechanism behind the specified correlation matrix. This group of samples can then be used as inputs, one-by-one, to broader and arguably real-world portfolio simulations. While this is certainly true, at best the group of sample correlation matrices, then, only provide indirect inferential value, with what is arguably a notable lack of control. For example, the group of samples cannot be used to specify, for controlled portfolio simulations, the two matrices representing

the 95% confidence interval under a given null hypothesis for the dependence structure, which is the kind of targeted, controlled capability needed for precise, powerful testing and consequent, targeted decision-making. Some may argue that the group of sample matrices can be used with ad hoc measures of 'distance' from a hypothesized matrix of 'true' values (e.g. a Euclidean 'norm' distance from, say, the identity matrix), yet such multivariate distances can be measured in many different and equally valid ways under various conditions (this is addressed in more detail in the Generalized Entropy chapter below). The same 'distances' also can have different interpretations under different conditions, and even widely used ones can be 'wrong' when applied to very commonly used dependence measures in this setting (see Zhang et al., 2024, for a compelling example). So making inferences based on them remains arguably as ad hoc, at best, as the arbitrary choice of how to measure distance between a sample matrix and its null hypothesis. Neither can such distances be used to rank order the sampled matrices to obtain, say, an empirical cdf, because different distances will yield different rank orderings. The only 'distance' that avoids these issues is probability itself, most conveniently represented as a cdf value.

In the end, for real inferential capability and consequential decision-making ability, what is necessary here is an analytically rigorous connection between a specific sampled correlation matrix and its associated, properly defined cdf value, and none of these sampling methods provide this. Fortunately, NAbC does, as is discussed further below. But first, I finish reviewing results from the extant literature that cover 1. distributional derivations of Pearson's matrix, but under the more general case of realistic, financial returns data, as well as 2. sample generation algorithms of Pearson's matrix under these same conditions (which should correspond to their stylized, empirical characteristics). For 1., I cover three recent and intriguing methods, labelled A.-C. below.

Distributional Results, More General Conditions

A. Archakov & Hansen (2021) introduce an original parameterization of Pearson's matrix that maps uniquely, one-to-one, to the positive definite space, thus providing a density for inference. It is valid under general conditions, based on the Fisher Z transformation, remains invariant to reorderings of the variables/assets, i.e. the rows and columns of the matrix, and is accompanied by an algorithm that provides the inverse mapping from the parameterization to the correlation matrix (i.e. a matrix level quantile function).

This approach is unique, but the method still has limitations. As the authors state, "This makes the transformation potentially useful for ... inference. These attributes tend to deteriorate as C approaches singularity. This is not unexpected, because it is also true for the Fisher transformation when the correlation is close to ±1." As previously noted, Taraldsen (2021) similarly shows that the approximate density of the pairwise correlation using Fisher's Z-transformation loses accuracy as correlation values approach ±1, especially for smaller samples. This is consistent with the authors' comments here, but they state this may only be material under extreme conditions. All else equal, having a method that avoided this non-robustness issue altogether would be preferrable, especially because such correlation breakdowns are exactly when we most need robust, accurate inferences.

Also, the method only provides the distribution of the entire correlation matrix: it does not appear to be able to modify correlation matrices, cell-by-cell, probabilistically, based on their individual distribution, for things like scenarios and 'what if' analyses. While this may not be a stated objective of the method, all else equal it would be an incredibly useful feature for stress testing and scenario analysis, as well as attribution analyses. The same holds true for larger submatrices of the matrix, i.e. submatrices larger than one cell, representing one pairwise association. Note that NAbC shares none of these limitations, but shares all of the method's advantages listed above, in addition to many others.

B. Lan et al. (2020) take a fully Bayesian approach to this problem for both covariance and Pearson's correlation matrices. Similar to NAbC, they use the Cholesky factorization to automatically enforce positive definiteness, and by defining distributions on spheres as NAbC does, utilize a large class of flexible prior distributions. This method includes *estimation* of the correlation/covariance matrix, which NAbC does not as described above, but it also lacks very important features that NAbC provides. As the authors state, "The priors for correlation matrix specified through the sphere-product representation are in general dependent among component variables. For example, the method we use to induce uncorrelated prior between y_i and y_j $(i < j)$ by setting $l_{jk} \approx 0$ for $k \leq i$ has a direct consequence that $\text{Cor}(y'_i, y_j) \approx 0$ for $i' \leq i$. In another word, more informative priors (part of the components are correlated) may require careful ordering in $\{y_i\}$. To avoid this issue, one might consider the inverse of covariance (precision) matrices instead. This leads to modeling the conditional dependence, or *Markov network* ... Our proposed methodology applies directly to (dynamic) precision matrices/processes, which will be our future direction."

Fortunately, NAbC does not suffer from this order-dependence problem. Like Archakov & Hansen (2021) its results are invariant to the ordering of the rows and columns of the matrix, but unlike Archakov & Hansen (2021) or Lan et al. (2020) it can 'freeze' any submatrix of the correlation matrix, even if it is non-contiguous, as dictated by any particular scenario or stress test, and still obtain a valid, finite-sample distribution for the matrix. There are no unintended 'dependencies' between cells that confound these results, whether the matrix is restricted or entirely unrestricted. As discussed further below, the 'unintended dependencies' problem is one that other researchers have struggled with, but which NAbC avoids altogether.

C. Like Lan et al. (2020), Ghosh et al. (2021) also take a fully Bayesian approach to this problem, and just like NAbC, they reparameterize Cholesky factors in terms of hyperspherical coordinates where the angles vary freely in the range [0, π). Their focus is on estimation, although as a Bayesian approach it is comprehensive and provides credible regions. Among its arguable limitations, however, is that its use is restricted to parametric priors, which given the non-small dimensions of most financial portfolios (e.g. Bongiorno & Challet (2023a) call p=100, which has p(p-1)/2=4,950 pairwise cells, 'mid-sized') it is hard to see how this would not limit its implementation under complex, real world financial data conditions (i.e. multivariate distributions of dimension p=100+ with margins with different and varying degrees of serial correlation, asymmetry, (non)stationarity, and heavy-tailedness). In other words, it is hard to imagine a

fully parametric multivariate distribution of dimension p=100 or greater that was analytically tractable but simultaneously able to adequately incorporate all of the characteristics listed above, regardless of the shrinkage or selection priors used. In contrast, NAbC makes use of flexible nonparametric kernels that fit ANY angles distribution resulting from ANY data generating mechanism (with finite mean and variance for Pearson's matrix). Also, like A. and B. above, the approach of Ghosh et al. (2020) approach does not appear to have the capability of modeling submatrices while leaving select cells of the correlation matrix 'untouched.' This is absolutely essential for flexible and realistic scenario modeling and (reverse) stress testing, and one of the many advantages NAbC provides. Now I treat some of the more recent, general-case sample generation algorithms, listed as D.-F. below.

Sampling Algorithms, More General Conditions

D. Marti (2019) proposes using generative adversarial networks (GANs) to incorporate the stylized empirical characteristics of financial portfolios' correlation matrices into an algorithm that directly generates samples of the all-pairwise matrix (CorrGAN), as opposed to samples of returns. This appears to be the first method to attempt this approach. The stylized characteristics include positive-shifted correlations, Marchenko-Pastur distributed correlations excepting a few large eigenvalues, Perron-Frobenius property (positive entries of the first eigenvector), hierarchical correlation structure, and scale-free property of the corresponding minimum spanning tree. Marti (2019) does not address how computationally intensive is the method, but apparently it is not prohibitively so as he implements it on 100x100 matrices in a follow-on blog post (see Marti, 2020).

There are three main limitations to this approach. First, as the author notes, while it appears to capture most of the identified distributional stylized facts, it does not capture the tails well. This arguably is the most important part of the distribution, as it is critically related to portfolio risk analytics, and many if not most scenarios, especially those that incorporate events like correlation breakdowns. In addition to trading algorithms, these are the stated purposes of the method, so difficulty estimating the tails of the distribution is not a minor limitation. Secondly, the method generates samples that "are not exactly correlation matrices" with non-unit diagonals and negative eigenvalues. Marti (2019) states that positive definiteness is enforced ex post using Higham (2002). I have used Higham (2002) extensively in my research in this setting, closely examining its effects on both the spectral distribution and the distribution of the correlation matrix itself, and have found that both can be dramatically distorted when Higham (2002) is used.[26] However, simply discarding non-positive definite samples is not the answer as this, too, will distort its true distribution. The only way to simulate the matrix and retain inferential validity, at least as it is affected by its eigen decomposition, is to use a method that automatically enforces positive definiteness, ex ante. Finally, the last limitation is that the samples generated by the method do not

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[26] To be clear, this is not a critique of Higham (2002), which is seminal and extremely useful in wide-ranging, applied settings. Rather, it is only to say that 'fixing' non-positive definite matrices that are generated by non-trivially complex algorithms often, if not usually, strongly distorts the **distribution** of the sample matrices, as well as the associated spectral distribution. This is readily verified empirically.

retain inferential validity generally, that is, they are not associated with a probability of occurrence or a cdf value, as described above.

E. Papenbrock et al. (2021) develop a novel and intriguing approach to simulating correlation matrices for financial markets using evolutionary algorithms. These allow for the flexible yet robust incorporation of many observed features of real-world financial correlation matrices (the list is similar to that of Marti (2019), with some enrichments). The algorithm scales well and can be used for backtesting, pricing, and hedging correlation-dependent investment strategies and financial products. However, it has several limitations: the first relates to how upper and lower barriers are established for the sampled correlation matrices, which the authors describe as "This neighborhood could be defined in a static way or by expert knowledge." Regarding the latter option, making this criterion (strictly) subjective arguably defeats the purpose of objective, quantitative analysis in this setting. Regarding the former option, Papenbrock et al. (2021) suggest using the most extreme values of the matrices, although none of the implementations listed (e.g. random matrix denoising, shrinkage, or exponential weighting) are inferentially valid in themselves, that is, they do not allow for probabilistic inference. Arguably, if the range of the matrices sampled needs to be restricted at all, it should be restricted based on rigorously defined probabilistic bounds, say, 99% confidence intervals. Fortunately, this is a capability that NAbC provides.

The second limitation of Papenbrock et al. (2021) is shared with Marti (2019) in that the algorithm does not enforce positive definiteness ex ante. The authors do acknowledge the importance of positive definiteness in this setting, but do not explain how their algorithm handles non-positive definite samples. Again, both ignoring/eliminating them from consideration, and/or 'fixing' them with algorithms like that of Higham (2002), distort the distribution of the correlation matrix in non-trivial ways, and thus invalidate inferences based on it. Finally, the sample correlation matrices generated by the evolutionary algorithms are not inferentially valid in themselves, i.e. each is not associated with a cdf probability. Again, none of these limitations – subjective or ad hoc restriction of the sample space, or ex post enforcement of positive definiteness, or lack of inferential validity – apply to NAbC.

F. A sophisticated and more recent attempt at directly generating sample correlation matrices with stylized, real-world characteristics is that of Kubiak et al. (2024). They develop denoising diffusion probabilistic models (DDPMs) that, across multiple asset classes and market regimes, compare favorably against a number of alternate modern approaches, including CorrGAN of Marti (2019) and other GANs approaches,[27] variational autoencoders, and more traditionally, block bootstraps. Limitations of the approach are similar to those of the other 'direct sampling' algorithms: i. the matrices generated are not true correlation matrices, lacking unit diagonals and true asymmetry; ii. the matrices are not guaranteed to be positive definite, and when they are not, positive definiteness is enforced ex post using Higham (1988); and iii. the sampled matrices do not retain inferential validity, that is, they are not associated with a probability of occurrence or a cdf value, as described above.

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[27] Kubiak et al. (2024) believe this is due to "the standard instability issues commonly associated with GAN training. (p.5)"

This is an interesting area of research and the work of these approaches, real limitations notwithstanding, is quite encouraging. This is especially true because NAbC, which shares none of these limitations, can be applied to the realistic groups of samples that they generate to give them inferential validity (as long as the samples are not distorted, i.e. are representative of the true distribution). As long as researchers can find a way for these algorithms to generate true correlation matrices and automatically enforce positive definiteness ex ante, or convincingly prove that deviations from either are truly numerically de minimis along any dimension of analysis (which certainly has not been done to date), NAbC can be applied to their samples to provide inferential validity, that is, to associate a cdf value with each and every sample matrix generated so that the distribution can be used inferentially. Again, the starting point for NAbC's application is the known or well-estimated dependence matrix, and the known or well-estimated data generating mechanism (in these cases, the *matrix* generating mechanism), and these methods provide both (again, as long as the samples are representative of the true distribution). But until these two 'fixes' can be applied, with proof that this truly has been achieved numerically, if not analytically, real inferential challenges will remain for this path of 'direct matrix simulation' research.

It is notable that none of this work has demonstrated that generating samples of correlation/dependence matrices by first generating synthetic *returns* data that have all of the empirical, stylized characteristics of actual *returns* data, is not sufficient, if carefully done, to generate the desired sample matrices, *even* if these are not-historically-realized but rather, for plausible future scenarios. This connection needs to be established, mathematically and explicitly, because in reality the sample matrices are only and exclusively based on the sample returns, and without mathematically defining the path from the returns data to the stylized sample matrices, something is missing, if not wrongly specified, on one end or the other. It is not that we cannot or should not jump right to directly sampling the correlation/dependence matrix per se; only that the connection between all of the stylized characteristics of the correlation/dependence matrix, and those of the returns data on which it is based, needs to be rigorously established if we are to have real insight into the mechanics, provably appropriate simulations, and distribution-based inferences (if not causal drivers) of the former.

In the absence of an explanation as to why it is not preferable to start with the returns data itself (aside from computational considerations), I hypothesize that part of the motivation of taking the 'direct simulation' route, even if not explicitly stated, is the right-minded desire to separate and isolate the distribution of the correlation/dependence matrix from other characteristics of the distribution of the returns data. There obviously have been many approaches to accomplish this, and it can remain very challenging in certain circumstances, but this is exactly what NAbC does, fully inferentially, transparently, and in some cases fully analytically, as I show below. I start with a narrow but foundational case under restricted conditions, and then expand to fully general conditions, because the first provides a useful referential baseline for understanding the latter.

Correlations to Angles, Angles to Correlations

I continue with Pearson's for the first derivation and implementation of NAbC, and the data and correlation structure I initially presume is Gaussian data under no correlation: that is, Pearson correlation values of zero off the diagonal of the matrix as below.

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

identity matrix = for p = 4 assets

The key to the NAbC approach rests in its use of the spherical ANGLE, $\boldsymbol{\theta}$, between the two mean-centered data vectors of X and Y, as opposed to directly and only using of the values of the correlations themselves. As mentioned above, using angles to understand the distribution of Pearson's goes back at least to Fisher (1915), and it turns out to be a much more general framework applicable to any dependence measure whose all-pairwise matrix is positive definite, not just Pearson's. But to start with Pearson's, for a single pair of variables, providing a single bivariate correlation value, the relationship between angle value and correlation value is most readily seen in the widely known "cosine similarity," where the cosine of the angle equals the inner product divided by the product of the two vectors' (Euclidean) norms as in (16), which I show again below for the reader's convenience.

$$(16) \quad \cos\left(\hat{\theta}\right) = \frac{\text{inner product}}{\text{product of norms}} = \frac{\langle \mathbf{X}, \mathbf{Y} \rangle}{\|\mathbf{X}\|\|\mathbf{Y}\|} = \frac{\sum_{i=1}^{N}\left(X_i - \sum_{j=1}^{N} X_j\right)\left(Y_i - \sum_{j=1}^{N} Y_j\right)}{\sqrt{\sum_{i=1}^{N}\left(X_i - \sum_{j=1}^{N} X_j\right)^2}\sqrt{\sum_{i=1}^{N}\left(X_i - \sum_{j=1}^{N} Y_j\right)^2}} = \frac{Cov_{X,Y}}{s_X s_Y} = r_{X,Y}, \ 0 \leq \hat{\theta} \leq \pi$$

If a portfolio has p assets, the number of its pairwise relationships is npr=p(p-1)/2. For all these npr relationships, the matrix analogue to (16), as long as the matrix is symmetric positive definite,[28] is well established in the literature (Joarder and Ali, 1992, Pinheiro and Bates, 1996, Rebonato and Jackel, 2000, Rapisarda et al., 2007, Pouramadi and Wang, 2015, and Cordoba et al., 2018) and shown below, formulaically in (20)-(22) and in computer code (SAS/IML) in Table A. The steps for translating between correlations and angles, in both directions, are shown in A.-C. below.

A. estimate the correlation matrix from sample data
B. obtain the Cholesky factorization of the correlation matrix
C. use inverse trigonometric and trigonometric functions on B. to obtain corresponding spherical angles

and in reverse:

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[28] Note again that this is true not only for Pearson's, but also for all relevant dependence measures in this setting, as will be discussed below.

C. start with a matrix of spherical angles

B. apply trigonometric functions to obtain the Cholesky factorization

A. multiply B. by its transpose to obtain the corresponding correlation matrix

(see Rebonato & Jaeckel, 2000, Rapisarda et al., 2007, and Pourahmadi & Wang, 2015, but note a typo in the formula in Pourahmadi & Wang, 2015, for the first 3 steps)

Central to this correlation-angle translation mechanism is obtaining the Cholesky factor of the correlation/dependence matrix, which is usually a hard-coded function in most statistical and mathematical software. The relevant formulae are included below for completeness.

(20) A correlation matrix R will be real, symmetric positive-definite,[29] so the unique matrix B that satisfies $R = BB^T$ where B is a lower triangular matrix (with real and positive diagonal entries), and B^T is its transpose, is the Cholesky factorization of R. Formulaically, B's entries are as follows:

$$B_{j,j} = (\pm)\sqrt{R_{j,j} - \sum_{k=1}^{j-1} B_{j,k}^2}, \qquad B_{i,j} = \frac{1}{B_{j,j}}\left(R_{i,j} - \sum_{k=1}^{j-1} B_{i,k} B_{j,k}\right) \text{ for } i > j$$

The Cholesky factor can be viewed as a matrix analog to the square root of a scalar, because like a square root the product of it and its transpose yields the original matrix. Importantly, the Cholesky factor places us on the UNIT hyper-(hemi)sphere (where scale does not matter) because the sum of the squares of its rows always equals one. Next, we recursively apply inverse trigonometric and trigonometric functions to each cell of the Cholesky factor to obtain each cell's angle value; or in reverse, we obtain a correlation/dependence value from each cell's angle value (see both Joarder & Ali, 1992, and Rapisarda et al., 2007, for meticulous derivations of these formulas). Note that this relationship is one-to-one, with a unique correlation/dependence matrix yielding a unique angles matrix, and vice versa.

(21) For R, a p x p correlation matrix,

$$R = \begin{bmatrix} 1 & r_{1,2} & r_{1,3} & \cdots & r_{1,p} \\ r_{2,1} & 1 & r_{2,3} & \cdots & r_{2,p} \\ r_{3,1} & r_{3,2} & 1 & \cdots & r_{3,p} \\ r_{4,1} & r_{4,2} & r_{4,3} & \cdots & r_{4,p} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ r_{p,1} & r_{p,2} & r_{p,3} & \cdots & 1 \end{bmatrix},$$

---

[29] Semi-positive definiteness includes the case of eigenvalues exactly equal to zero, which I largely ignore herein as a border case relevant mainly for textbook examples.

$R = BB^t$ where B is the Cholesky factor of R and

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \cos(\theta_{2,1}) & \sin(\theta_{2,1}) & 0 & \cdots & 0 \\ \cos(\theta_{3,1}) & \cos(\theta_{3,2})\sin(\theta_{3,1}) & \sin(\theta_{3,2})\sin(\theta_{3,1}) & \cdots & 0 \\ \cos(\theta_{4,1}) & \cos(\theta_{4,2})\sin(\theta_{4,1}) & \cos(\theta_{4,3})\sin(\theta_{4,2})\sin(\theta_{4,1}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \cos(\theta_{p,1}) & \cos(\theta_{p,2})\sin(\theta_{p,1}) & \cos(\theta_{p,3})\sin(\theta_{p,2})\sin(\theta_{p,1}) & \cdots & \prod_{k=1}^{n-1}\sin(\theta_{p,k}) \end{bmatrix}$$

for $i > j$ angles $\theta_{i,j} \in (0, \pi)$.

To obtain an individual angle $\theta_{i,j}$, we have:[30]

For $i > 1$: $\quad \theta_{i,1} = \arcos(b_{i,1})$ for $j=1$; and $\theta_{i,j} = \arcos\left(b_{i,j} \Big/ \prod_{k=1}^{j-1}\sin(\theta_{i,k}) \right)$ for $j > 1$

(22) To obtain an individual correlation, $r_{i,j}$, we have, simply from $R = BB^T$:

$$r_{i,j} = \cos(\theta_{i,1})\cos(\theta_{j,1}) + \prod_{k=2}^{i-1}\cos(\theta_{i,k})\cos(\theta_{j,k})\prod_{l=1}^{k-1}\sin(\theta_{i,l})\sin(\theta_{j,l}) + \cos(\theta_{j,i})\prod_{l=1}^{i-1}\sin(\theta_{i,l})\sin(\theta_{j,l}) \text{ for } 1 \le i < j \le n$$

SAS/IML code translating correlations to angles and angles to correlations is shown in Table A below:

---

[30] Note that a similar recursive relationship exists between partial correlations, although its sample-generating algorithm (see Madar, 2015) does not generalize beyond Pearson's correlations, i.e. to all positive definite measures of dependence, as does that of NAbC, as shown below.

TABLE A:

<table>
<tr><th>Correlations to Angles</th><th>Angles to Correlations</th></tr>
<tr><td>

```
* INPUT rand_R is a valid correlation matrix;

cholfact = T(root(rand_R, "NoError"));

rand_corr_angles = J(nrows,nrows,0);
  do j=1 to nrows;
    do i=j to nrows;
      if i=j then rand_corr_angles[i,j]=.;
      else do;
        cumprod_sin = 1;
        if j=1 then rand_corr_angles[i,j]=arcos(cholfact[i,j]);
        else do;
          do kk=1 to (j-1);
            cumprod_sin = cumprod_sin*sin(rand_corr_angles[i,kk]);
          end;
          rand_corr_angles[i,j]=arcos(cholfact[i,j]/cumprod_sin);
        end;
      end;
    end;
  end;

* OUTPUT rand_corr_angles is the corresponding matrix of angles;
```

SAS/IML code (v9.4)

</td><td>

```
* INPUT rand_angles is a valid matrix of correlation angles;

Bs=J(nrows, nrows, 0);
do j=1 to nrows;
  do i=j to nrows;
    if j>1 then do;
      if i>j then do;
        sinprod=1;
        do gg=1 to (j-1);
          sinprod = sinprod*sin(rand_angles[i,gg]);
        end;
        Bs[i,j]=cos(rand_angles[i,j])*sinprod;
      end;
      else do;
        sinprod=1;
        do gg=1 to (i-1);
          sinprod = sinprod*sin(rand_angles[i,gg]);
        end;
        Bs[i,j]=sinprod;
      end;
    end;
    else do;
      if i>1 then Bs[i,j]=cos(rand_angles[i,j]);
      else Bs[i,j]=1;
    end;
  end;
end;
rand_R = Bs*T(Bs);

* OUTPUT rand_R is the corresponding correlation matrix;
```

</td></tr>
</table>

The above all is well-established and straightforward,[31] and demonstrates, as we know intuitively, that **scale does not (and should not) matter when it comes to dependence measures;**[32] again, in this setting, this is because geometrically, the Cholesky factor places us on the UNIT hyper-(hemi)sphere. Importantly, the Cholesky factor also ensures that sampling based directly on the resulting angles will yield only positive definite matrices, as the Cholesky factor remains undefined otherwise. This **automatic enforcement of positive definiteness makes this approach much more efficient than others** that require ex post verification of positive definiteness, and subsequent resampling or enforcement when this requirement is violated (examples of this discussed above include Makalic & Schmidt, 2018; Cordoba et al. 2018; Marti, 2019; Papenbrock et al., 2021; and Kubiak et al., 2024). This

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[31] Reliance on spherical angles and hypersphere parameterizations is increasingly common in quantitative finance (see for some examples Li, Q., 2018; Helton, 2020; Golts & Jones, 2009; Zhang, 2022; Saxena et al., 2023; and Zhang & Yang, 2023), in large part due to its scale invariance: it has even been used to define entire financial markets (see Kim and Lee, 2016).

[32] Scale invariance is proved and widely cited for Pearson's, Kendall's, and Spearman's (see Xu et al., 2013, and Schreyer et al., 2017 examples).

inefficiency grows very rapidly with the size of the matrix/portfolio, as shown in the ratio below in (23) (see Bohn and Hornik, 2024, and Pourahmadi & Wang, 2015).

$$(23) \quad \Pr\left(rand\,"R" \sim PosDef\right) = X = \frac{\prod_{j=1}^{p-1}\left[\sqrt{\pi}\,\Gamma\left(\frac{j+1}{2}\right)\right]^{j}}{2^{p(p-1)/2}} < \prod_{j=1}^{p-1}\left[\frac{\sqrt{\pi}}{2}\right]^{j} = \left[\frac{\sqrt{\pi}}{2}\right]^{p(p-1)/2} ; \lim_{p\to\infty}\left[X\right] = 0$$

Even for relatively small matrices of dimension p=25, the odds of successfully randomly generating a single valid positive definite correlation matrix, by uniformly sampling the off-diagonal correlation values themselves across values ranging from –1.0 to 1.0, are less then 2 in 10 quadrillion, leading to prohibitively inefficient sampling. Consequently, even when sampling-rejection algorithms achieve some efficiency gains, realistically the sampling approach in this setting should possess automatic enforcement of positive definiteness, ex ante. Conceptually, an imperfect but apt analogy is to a rubiks cube: the colored stickers on the cube cannot simply be peeled off and repasted, even some of the time, to solve the cube. The valid solution must be obtained by (always) following the rules governing shifts in the cube, and every move of each of the small cubes (correlation cells) affects the positions of many of the other cubes (correlation cells), not just the one we need to reposition. Similarly with sampling the correlation/dependence matrix: converting to the Cholesky factor (en)forces positive definiteness by forcing the matrix onto the UNIT hyper-(hemi)sphere, where we can subsequently use the distributions of the angles to perturb it and obtain, after re-translation, the distribution of the original correlation/dependence matrix, without violating positive definiteness. This is done simply by following steps A., B., and C., and C., B., and A., above.

Another crucial characteristic of these angles is that **they are random variables whose multivariate relationship is one of independence** (see Pourahmadi & Wang, 2015, Tsay & Pourahmadi, 2017, and Ghosh et al., 2020). This is critically important for practical usage as it enables the straightforward construction of the multivariate distribution of a matrix of angles, which is the more important objective here (vs merely sampling) and essential for the application of NAbC below.

Finally and most critically, the above demonstrates that **the angles between pairwise data vectors contain ALL the information that exists regarding dependence between the two variables** because the only information we lose by translating to the unit hyper(hemi-)sphere is scale (see Fernandez-Duren & Gregorio-Dominguez, 2023, and Zhang & Songshan, 2023, as well as Opdyke, 2024a). This will be covered more extensively below.

So with all this in mind we proceed with the use of the angles as described and defined above. The goal is to use the angles as the basis for 1. sample generation of the correlation/dependence matrix; and more importantly, 2. definition of the multivariate distribution of the correlation/dependence matrix.

<u>Fully Analytic Angles Density, and Efficient Sample Generation</u>

Once we have the matrix of angles (per (21) and Table A above), one angle for each value in the all-pairwise correlation/dependence measure matrix, we use the well-established finding that, to sample

uniformly from the space of positive definite matrices, the probability density function (pdf) must be proportional to the determinant of the Jacobian of the Cholesky factor (24) (see Cordoba, 2018, Pourahmadi & Wang, 2015, Lewandowski et al., 2009).

(24) $$\det\left[J(U)\right] = 2^p \prod_{i=1}^{p-1} u_{ii}^i \ \text{ where } U \text{ is the Cholesky factorization of correlation matrix } R = UU^t$$

We see directly from (24) that $\sin^k(x)$, suitably normalized in (25), satisfies this requirement (see Pourahmadi & Wang, 2015, and Makalic & Schmidt, 2018).

(25) $$f_X(x) = c_k \cdot \sin^k(x), \ x \in (0,\pi), \ k = 1,2,3,\ldots,(\#\text{columns}-1), \ \text{and } c_k = \frac{\Gamma(k/2+1)}{\sqrt{\pi}\,\Gamma(k/2+1/2)}$$

Although not explained in Makalic & Schmidt (2018), importantly note that k = #columns – column# (so for the first column of a p=10x10 matrix, k=9; for the second column, k=8, etc.).

Beyond (25), however, we need both the cumulative distribution function (cdf) and its inverse, the quantile function, to make use of this density for sampling and other purposes. The most widely used and straightforward method of sampling is inverse transform, whereby the values of a uniform random variate are passed to the quantile function to generate sampled values. Yet regarding the cdf corresponding to (25) above, Makalic & Schmidt (2018) state, "Generating random numbers from this distribution is not straightforward as the corresponding cumulative density [sic] function, although available in closed form, is defined recursively and requires O(k) operations to evaluate. The nature of the cumulative density [sic] function makes any procedure based on inverse transform sampling computationally inefficient, especially for large k."

Fortunately, that turns out not to be the case, as Opdyke (2020) derived an analytic, non-recursive expression of the cdf below in (26):

(26)

$$F_X(x;k) \sim \frac{1}{2} - c_k \cdot \cos(x) \cdot {}_2F_1\left[\frac{1}{2}, \frac{1-k}{2}; \frac{3}{2}; \cos^2(x)\right] \text{ for } x < \frac{\pi}{2},$$

$$\sim \frac{1}{2} + c_k \cdot \cos(x) \cdot {}_2F_1\left[\frac{1}{2}, \frac{1-k}{2}; \frac{3}{2}; \cos^2(x)\right] \text{ for } x \geq \frac{\pi}{2}$$

where the Gaussian hypergeometric function ${}_2F_1\left[a,b;c;r\right] = \sum_{n}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{r^n}{n!}$

where $(h)_n = h(h+1)(h+2)\cdots(h+n-1)$, $n \geq 1$, $(h)_0 = 1$, and $|r| < 1$, $c \neq 0,-1,-2,\ldots$

As mentioned in a footnote above, the Gaussian hypergeometric function makes many interesting appearances in this setting, but it is admittedly cumbersome mathematically. Yet Opdyke (2022, 2023,

and 2024) has shown that (26) can be simplified further, based on some arguably obscure hypergeometric identities in (27) below:

(27)

For $c = a + 1$ and $0 < r < 1$ simultaneously, which holds in this setting, we have $_2F_1[a,b;c;r] = B(r;a,1-b)\left(a/r^a\right)$

where $B(r;a,b) = \int_0^r u^{a-1}(1-u)^{b-1}\,du = $ the incomplete beta function (see DLMF, 2024)

In addition we have

$F_{Beta}(r;a,b) = B(r;a,b)/B(a,b)$ where $B(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = $ the complete beta function, so

$B(r;a,b) = F_{Beta}(r;a,b) \cdot B(a,b)$ (see Weisstein, E., 2024a and 2024b)

Combining terms we have

$$F_X(x;k) \sim \frac{1}{2} - c_k \cdot \cos(x) \cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right] \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+k}{2}\right)}{\Gamma\left(\frac{2+k}{2}\right)} \cdot \left([1/2]\big/\sqrt{\cos^2(x)}\right) \text{ for } x < \frac{\pi}{2},$$

$$F_X(x;k) \sim \frac{1}{2} + c_k \cdot \cos(x) \cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right] \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+k}{2}\right)}{\Gamma\left(\frac{2+k}{2}\right)} \cdot \left([1/2]\big/\sqrt{\cos^2(x)}\right) \text{ for } x \geq \frac{\pi}{2}$$

Recognizing that the complete Beta function is the inverse of the normalization factor of c(k) for these values, their product equals 1 and cancels, as do the two cosine terms, and we obtain the following signed beta cdf:

$$F_X(x;k) \sim \frac{1}{2} - \left(\frac{1}{2}\right) \cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right] \text{ for } x < \frac{\pi}{2},$$

$$\sim \frac{1}{2} + \left(\frac{1}{2}\right) \cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right] \text{ for } x \geq \frac{\pi}{2}$$

And now, with this straightforward, fully analytic, non-recursive cdf, we can obtain a straightforward, fully analytic quantile function of the angle distribution in (28):

(28)

Let $p = \Pr(x \geq X)$. Then for $x < \dfrac{\pi}{2}$,

$$p = \frac{1}{2} - \left(\frac{1}{2}\right) \cdot F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right]$$

$$-2p = -1 + F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right]$$

$$1 - 2p = F_{Beta}\left[\cos^2(x); \frac{1}{2}, \frac{1+k}{2}\right]$$

$$F_{Beta}^{-1}\left(1 - 2p; \frac{1}{2}, \frac{1+k}{2}\right) = \cos^2(x)$$

$$\sqrt{F_{Beta}^{-1}\left(1 - 2p; \frac{1}{2}, \frac{1+k}{2}\right)} = \cos(x)$$

$$\arccos\left(\sqrt{F_{Beta}^{-1}\left(1 - 2p; \frac{1}{2}, \frac{1+k}{2}\right)}\right) = x$$

(Note that arcos is arc-cosine, the inverse of the cosine function.)

We must reflect the symmetric angle density for p≥0.5, so we have

$$x = \arccos\left(\sqrt{F_{Beta}^{-1}\left(1 - 2p; \frac{1}{2}, \frac{1+k}{2}\right)}\right) \text{ for } p < 0.5,$$

$$= \pi - \arccos\left(\sqrt{F_{Beta}^{-1}\left(1 - 2[1-p]; \frac{1}{2}, \frac{1+k}{2}\right)}\right) \text{ for } p \geq 0.5$$

Importantly, although often ignored in the sample-generation literature (see, for example, Makalic & Schmidt, 2018), note that properly adjusting for sample size, n, and degrees of freedom gives $k \leftarrow k + n - \#cols - 2$, so consequently, $k = n - column\# - 2$.[33]

So now from (28) above we have for the angles distribution, under the Gaussian identity matrix, for the first time together, the pdf, cdf, and quantile function in (29):

(29)

$$f_X(x) = c_k \cdot \sin^k(x), \ x \in (0, \pi), \ k = 1,2,3\ldots\#columns - 1, \text{ and } c_k = \frac{\Gamma(k/2+1)}{\sqrt{\pi}\Gamma(k/2+1/2)}$$

---

[33] So notably, the angles distributions vary systematically based on their (column) position in the matrix, even though the distributions of the correlations themselves do not, as is discussed in later chapters.

$$F_X\left(x;k\right) \sim \frac{1}{2} - \left(\frac{1}{2}\right) \cdot F_{Beta}\left[\cos^2\left(x\right);\frac{1}{2},\frac{1+k}{2}\right] \text{ for } x < \frac{\pi}{2},$$

$$\sim \frac{1}{2} + \left(\frac{1}{2}\right) \cdot F_{Beta}\left[\cos^2\left(x\right);\frac{1}{2},\frac{1+k}{2}\right] \text{ for } x \geq \frac{\pi}{2}$$

$$F^{-1}\left(p;k\right) = \arccos\left(\sqrt{F_{Beta}^{-1}\left(1-2p;\frac{1}{2},\frac{1+k}{2}\right)}\right) \text{ for } p < 0.5;$$

$$= \pi - \arccos\left(\sqrt{F_{Beta}^{-1}\left(1-2[1-p];\frac{1}{2},\frac{1+k}{2}\right)}\right) \text{ for } p \geq 0.5$$

Apparently the first (and only other) presentation of this quantile function result comes from an anonymous blog post in March, 2018, although it was obtained via a different derivation, which serves to further validate the result.[34]

The above (29) now provides a fully analytic solution,[35] and in fact is so straightforward as to be readily implemented in a spreadsheet, and one is provided for download via the link below.

http://www.datamineit.com/JD%20Opdyke--The%20Correlation%20Matrix-Analytically%20Derived%20Inference%20Under%20the%20Gaussian%20Identity%20Matrix--02-18-24.xlsx

So contrary to the assertions of Makalic & Schmidt (2018), the straightforward approach of inverse transform sampling **can** be used in this setting, for this narrow case, to very efficiently generate samples from the correlation matrix. And in fact, this is the most efficient way to sample it. Roman (2023) has compared Makalic and Schmidt (2018) to the above method and obtained over 30% decrease in runtime when using Opdyke (2022, 2023, and 2024). But of course, when used analytically, for example via the linked spreadsheet, these results are instantaneous.

But sampling arguably is the less important of our two goals, because with a fully analytic finite-sample distribution, we can define, exactly for a given sample size, the p-value of a given cell, and the confidence interval of a given cell. The one-sided p-value simply is the CDF value for the lower tail, or [1 – (CDF

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[34] See Xi'an, March, 2018 (https://stats.stackexchange.com/questions/331253/draw-n-dimensional-uniform-sample-from-a-unit-n-1-sphere-defined-by-n-1-dime/331850#331850 and https://xianblog.wordpress.com/2018/03/08/uniform-on-the-sphere-or-not/). In the interest of proper attribution, a reference on the website to the book "The Bayesian Choice" hints that the Xi'an pseudonym is Christian Robert, a professor of Statistics at Université Paris Dauphine (PSL), Paris, France, since 2000 (https://stats.stackexchange.com/users/7224/xian).

[35] Note that I use the term 'analytic' as opposed to 'closed-form' because I am unaware of a closed-form algorithm for the inverse cdf of the beta distribution (see Sharma and Chakrabarty, 2017, and Askitis, 2017). However, for all practical purposes this is essentially a semantic distinction since this quantile function is hard-coded into all major statistical / econometric / mathematical programming languages.

value)] for the upper tail (30), and due to this pdf's symmetry, the two-sided p-value is simply two times either one-sided value. Correspondingly, the confidence interval for the critical value alpha is based on the quantile function as in (31)

(30) one-sided p-value = $F_X(x;k)$ or $1 - F_X(x;k)$ where k = n − column# − 2;

      two-sided p-value = 2 x one-sided p-value

(31) $F^{-1}(\alpha/2;k)$ and $F^{-1}(1-\alpha/2;k)$ where, for a 95% confidence interval for example, α = 0.05

Notably, because the angles distributions are independent, the density of the entire matrix is simply the product of the densities of all the cells. This means we can readily define the p-value and confidence intervals of the entire matrix such that they are analytically consistent with those of the cells, because they are determined based directly on the cell level p-values and confidence intervals, respectively, as shown below.

      <u>Matrix-level p-values and Confidence Intervals</u>

As mentioned above, a key characteristic of the angles is that they are independent random variables, which makes defining their multivariate distribution straightforward: it is simply the product of all the angles' pdf's. But what does this mean for the p-value and confidence intervals for the entire matrix? Recall that a p-value is simply the probability of observing, based on a given data sample, a statistic value at least as large as what was observed, assuming the null hypothesis is true. The p-values defined above for each correlation/dependence cell are the probabilities of observing, for a given sample, angle values as large as what is observed assuming the null hypothesis is true. The fact that the angles are independent random variables, i.e. each is independent vis-à-vis all the other angles, makes obtaining the p-value for the entire matrix very straightforward. Note that the probability that <u>none</u> of the correlation/dependence cells are as extreme as what was observed is simply the product of one minus every p-value, because they are independent. So the probability that <u>one or more</u> of the correlation/dependence cells are as extreme or more extreme than what was observed is simply one minus this value, shown in (32) below, and this is the p-value for the entire matrix.

(32) $\text{matrix (2-sided) } pvalue = \left[1 - \prod_{i=1}^{p(p-1)/2} \left(1 - p\text{-}value_i \right) \right]$ where $p\text{-}value_i$ is the 2-sided p-value.

Another way of conceptualizing this is that if the null hypothesis of just one of the matrix cells is not true, then the null hypothesis for the entire matrix is not true, and this is what the matrix-level p-value measures: the probability that <u>at least one</u> of the cell-level null hypotheses is not true. If instead of p-values we were using critical values in p(p-1)/2 hypothesis tests, this would be exactly consistent with controlling the familywise error rate (FWER) of the joint hypothesis including all the cells of the all-pairwise matrix.[36] And just as no other approach to estimating FWER would increase statistical power in

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[36] Note that this approach has been used in the literature for addressing very closely related problems (see Fang et al., 2024).

this case due to the independence of the angles distributions,[37] no other definition of the matrix-level p-value will have greater power for the same reason.

Similarly, calculation of the confidence interval for the entire matrix (33) is essentially the same as that of the p-value, but of course it is divided in half to account for each tail, and the root of the critical values is taken, rather than the product. Otherwise, the calculations are identical to obtain the critical alphas for these 'simultaneous confidence intervals.'

(33) $\alpha_{crit-simult-LOW} = \left(1 - \left[1 - \alpha/2\right]^{\left(1/\left[\,p(p-1)/2\,\right]\right)}\right)$ and $\alpha_{crit-simult-HIGH} = 1 - \alpha_{crit-simult-LOW}$

These critical alphas, when inserted in the quantile function (28) and applied to every cell, provide the two correlation matrices that define and capture, say, (1-alpha)=(1-0.05)=95% of randomly sampled matrices under the null hypothesis, which in this case is the identity matrix. Again, it is the independence of the angles that makes these simultaneous confidence intervals very straightforward to calculate.

Importantly, again note that because we derived the quantile (inverse cdf) function in (28) above, we can go in either direction regarding these results: we can specify a correlation matrix and, under the null hypothesis of the identity matrix, obtain its p-values, both for the individual cells and the entire matrix, simultaneously. We also can specify a matrix of cdf values and obtain its corresponding correlation matrix, which is extremely useful and straightforward when constructing both stress and reverse stress scenarios. Finally, we can use simultaneous confidence intervals to obtain the two correlation matrices that form the matrix-level confidence interval.

Note that all these calculations are included in the downloadable spreadsheet (link provided above), with visible formulae corresponding to each step of these calculations for full transparency. In the next section below I expand these results for Pearson's to apply to all data conditions, and all values of the null hypothesis (i.e. any values for the matrix, not just the identity matrix).

NAbC: Pearson's Correlation, Real-World Financial Data, Any Matrix Values

Currently, the extant literature does not provide analytic forms for the angles distributions under general conditions. Deriving these appears to be a non-trivial problem. Spectral (eigenvalue) distributions, which many researchers turn to in this setting, have been shown to vary dramatically when data is characterized by different degrees of heavy-tailedness (see Burda et al., 2004, Burda et al., 2006, Akemann et al., 2009; Abul-Magd et al., 2009, Bouchaud & Potters, 2015, Martin & Mahoney, 2018; Heiny and Yao, 2022, and Opdyke, 2024a), as well as by different degrees of serial correlation (see Burda et al., 2004, 2011, Hisakado and Kaneko, 2023, and Opdyke, 2024a), and the literature provides no general

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[37] Other approaches for calculating the FWER that rely on, for example, resampling methods (see Westfall and Young, 1993, and Romano and Wolf, 2016) exploit dependence structure to increase power; here, under independence, they would provide no power gain over the analogue to (32) because there is no dependence structure for them to exploit.

analytic form under general, real-world financial data conditions – certainly nothing that is analogous to convergence to the Marchenko-Pastur distribution under iid conditions (see Marchenko and Pastur, 1967),[38] let alone under even more general, non-iid conditions. If angles distributions are of similar complexity, then deriving their analytic form under general conditions, if possible, currently appears to be a non-trivial, unsolved problem.

However, this need not be a showstopper for our purposes, in part because angles distributions have many characteristics that make them useful here generally, and more useful specifically than spectral distributions in this setting, by multiple criteria, including structurally, empirically, and distributionally.

Structurally: Aggregation level becomes relevant and important here. For a given correlation matrix R there are many more angles than eigenvalues (i.e. p(p-1)/2 cells vs p eigenvalues, for a factor of (p-1)/2 more). As a matrix *approaches* singularity (non-positive definiteness (NPD)), which arguably is the rule rather than the exception for non-small investment portfolios, a much smaller *proportion* of angles distributions will approach degeneracy (i.e. minimum/maximum values of zero and **π**) than is true for eigenvalue distributions (where more values will wrongly fall below zero). Consequently, the overall construction of the correlation matrix via $R = BB^T$ generally will remain much more stable than one based on an eigen decomposition of $R = V\Lambda V^{-1}$ where V is a matrix with column eigenvectors and Λ is a diagonal matrix of the corresponding eigenvalues.

Empirically: If an angle distribution approaches degeneracy, most of its values typically will approach 0 or **π**. But the relevant trigonometric functions (sin and cos) of these values are stable, and will simply approach -1, 0, or 1. This makes $R = BB^T$ a relatively stable calculation empirically, even if it produces an R that is approaching NPD. In contrast, eigenvalue/vector estimations are more numerically involved compared to the application of simple trigonometric functions, and this, combined with the fact that empirically, their upper bound is not well-bounded (in the general case),[39] makes their computation comparatively less numerically stable as matrices approach NPD.

Distributionally: As shown graphically below under challenging, real-world financial data conditions, the angles distributions are relatively "well behaved," both in the general sense and relative to spectral distributions. They are relatively smooth and typically unimodal, and clearly bounded on $\theta \in (0, \pi)$. Spectral distributions, based on the same data, very often are spikey,[40] multimodal, and for

[38] Note that some exceptions to convergence to this celebrated distribution do exist (see Li and Yao (2018), Hisakado and Kaneko (2023), Heiny and Yao (2022), and Maltsev and Malysheva (2024) for examples).

[39] Even though the largest eigenvalue is known to follow the Tracy-Widom distribution under certain sets of conditions, under others it can diverge, with unbounded support (see for example Li, 2025). Even when the latter cases do not hold mathematically, empirically, in practice, the largest eigenvalue can become so large that it is essentially unbounded.

[40] In fact, one of the most commonly encountered covariance (correlation) matrices under real world financial data conditions is the spiked matrix (see Johnstone, 2001), where one or few eigenvalues dominate and the majority of eigenvalues are close to zero, i.e. not reliably estimated. This further demonstrates that spectral approaches are far too limited and limiting to effectively solve this problem under real-world conditions.

practical purposes, empirically unbounded (at least in higher dimensions), all of which translates into larger variances and less tail accuracy. Simply put, they typically are much more complex and challenging to estimate precisely and accurately compared to individual angles distributions for a given correlation matrix R under real-world financial data.

All of this adds up to a more robust and granular basis for inference and analysis when relying on angles distributions as opposed to spectral distributions. Even more important that these considerations, however, is the fact that spectral distributions simply are at the wrong level of aggregation for these purposes: they remain at the (higher) level of the p assets of a portfolio – NOT at the granular level of the p(p-1)/2 pairwise associations of that portfolio, which is where both the angles distributions, and those of the correlations/dependence measure values themselves, lie. Consequently, while potentially very useful for things like portfolio factor analysis, spectral analysis simply is too blunt a tool for our purposes here: we need to be able to make inferences and monitor processes and conduct (reverse) scenario analyses and customized stress tests on ALL aspects of the dependence structure measured by the all-pairwise correlation/dependence matrix, at the granular level at which it is defined. The specific need for this in scenario and reverse scenario analyses is covered in more detail below.

So given the useful characteristics of the angles distributions, not to mention the fact that they remain at the right level of aggregation for granular analysis of the correlation/dependence matrix, we are able to proceed WITHOUT their analytic derivation. Rather, we can use a time-tested nonparametric approach, such as kernel estimation, to reliably define them. All this requires is a single simulation (say, N=10,000) based on the known or well-estimated correlation/dependence matrix, and its known or well-estimated data generating mechanism. These are the two stated requirements for the application of NAbC under general conditions. Then, after translating all N simulated correlation matrices to N matrices of angles, we fit a kernel to each empirical angle distribution, i.e. the empirical distribution of each angle for each cell of the matrix. We now have not only the distributions of all the individual angles, but also the multivariate distribution of the matrix, which is just the product of all the individual distributions due to their independence. Note that this goes in both directions: we can perform 'look-ups' on the empirically defined distribution to obtain the cdf value(s) corresponding to particular angle value(s), or use cdf value(s) to 'look up' corresponding angle (quantile) value(s). The kernel fitting smooths this empirical density to all (continuous) values. This process is described step by step below.

1. Simulate samples (say, N=10k) based on the specified/known or well estimated all-pairwise correlation/dependence matrix and the specified/known or well estimated data generating mechanism.
2. Calculate the corresponding N all-pairwise correlation/dependence matrices, and their Cholesky factorizations, and transform each of these into a lower triangle matrix of angles (as described above in (21)).
3. Fit kernel densities to each of the p(p-1)/2 empirical angle distributions, each having N observations.

4. Generate samples (say, N=10k) based on the densities in 3.[41]
5. Convert the samples from 4. back to a re-parameterized Cholesky factorization, and then multiply by its transpose to obtain a set of N validly sampled correlation/dependence matrices (as described above in (22)). Positive definiteness is enforced automatically as the Cholesky factor places us on the **unit** hyper-hemisphere. All sample generation hereafter uses just 4. And 5.

The distribution of correlation matrices from 5. is identical to that of 2., but after the kernel densities are fit once in 3., generating samples in 4. is orders of magnitude faster than relying on direct simulations in steps 1. and 2. So one simulation gives us the distribution of each and every angle, corresponding to each and every correlation/dependence cell. And now going forward using 4.-5., rather than 1. and 2., allows for correct probabilistic inference, both at the cell level and at the matrix level, due to the independence of the angles distributions (remember the correlations themselves are NOT independent, so 1. and 2. provide no direct inferential capability). This reliance on the angles, and their subsequent transformation to correlations, allows us to isolate specifically the distribution of the entire correlation/dependence matrix, for probabilistic inference, without touching any other distributional aspect of the data, which is the point of the methodology. Of course, either a direct simulation (step 1. above) or a cavalier 'bootstrap' of correlation/dependence matrices based on step 1. fails at this objective, because the non-independence of the cells undermines the validity of any empirically-based inference based on simple metrics (e.g. distances) of each sample within the group of sample matrices. In other words, direct simulation does not preserve inferential capabilities, but angles simulation does.

So this framework is essentially identical to that for the specific case of the Gaussian identity matrix, with the only difference being it is based on nonparametrically defined, as opposed to parametrically defined, angles distributions. Before covering implementation details below, I show some examples of graphs of the angles distributions and the corresponding spectral distribution under challenging, simulated financial returns data. The multivariate returns distribution of the portfolio is generated based on the t-copula of Church (2012), with p=5 assets, varying degrees of heavy-tailedness (df=3, 4, 5, 6, 7), skewness (asymmetry parameter=1, 0.6, 0, -0.6, -1), non-stationarity (standard deviation=3σ, σ/3, σ; n/3 observations each), and serial correlation (AutoRegressive1=-0.25, 0, 0.25, 0.50, 0.75), with a block correlation structure shown in (34) below and n=126 observations for a half year of daily returns.[42] The spectral distribution is compared against Marchenko-Pastur as a referential baseline.

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[41] Algorithms for sample generation based on commonly used kernels (e.g. the Gaussian and Epanechnikov) are widely known. An example of the latter is simply the median of three uniform random variates (see Qin and Wei-Min, 2024).

[42] Note that this is only approximately Church's (2012) copula, which incorporates varying degrees of freedom (heavy-tailedness) and asymmetry, because I also impose serial correlation and non-stationarity on the data (and subsequently empirically rescale the marginal densities).

1	-0.3	-0.3	0.2	0.2
-0.3	1	-0.3	0.2	0.2
-0.3	-0.3	1	0.2	0.2
0.2	0.2	0.2	1	0.7
0.2	0.2	0.2	0.7	1

(34)

Graph 1a: Spectral Distribution–Angles/Kernel Perturbation v Data Simulations v Marchenko Pastur

Eigenvalue Distributions -- DGM vs. Angles (epan-SILVR09-15): DGM=MVTVNS, Mat=5x5, #Obs=126, #Sims = 10000
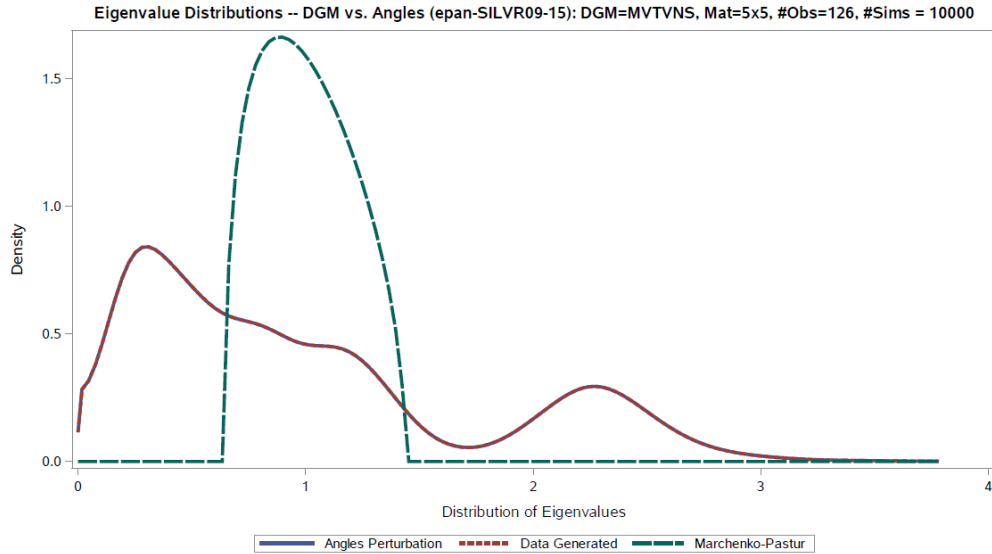
Graphs 1-10: Angles Distributions – Angles/Kernel Perturbation v Data Simulations v Independence

Emp Angles Dists v Sin(x)^k(k=1)[row=5, Angle#=1]: DGM=MVTVNS (epan-SILVR09-15), Mat=5x5, #Obs=126, #Sims = 10000

Emp Angles Dists v Sin(x)^k(k=2)[row=4, Angle#=2]: DGM=MVTVNS (epan-SILVR09-15), Mat=5x5, #Obs=126, #Sims = 10000

Emp Angles Dists v Sin(x)^k(k=2)[row=5, Angle#=3]: DGM=MVTVNS (epan-SILVR09-15), Mat=5x5, #Obs=126, #Sims = 10000

Emp Angles Dists v Sin(x)^k(k=3)[row=3, Angle#=4]: DGM=MVTVNS (epan-SILVR09-15), Mat=5x5, #Obs=126, #Sims = 10000
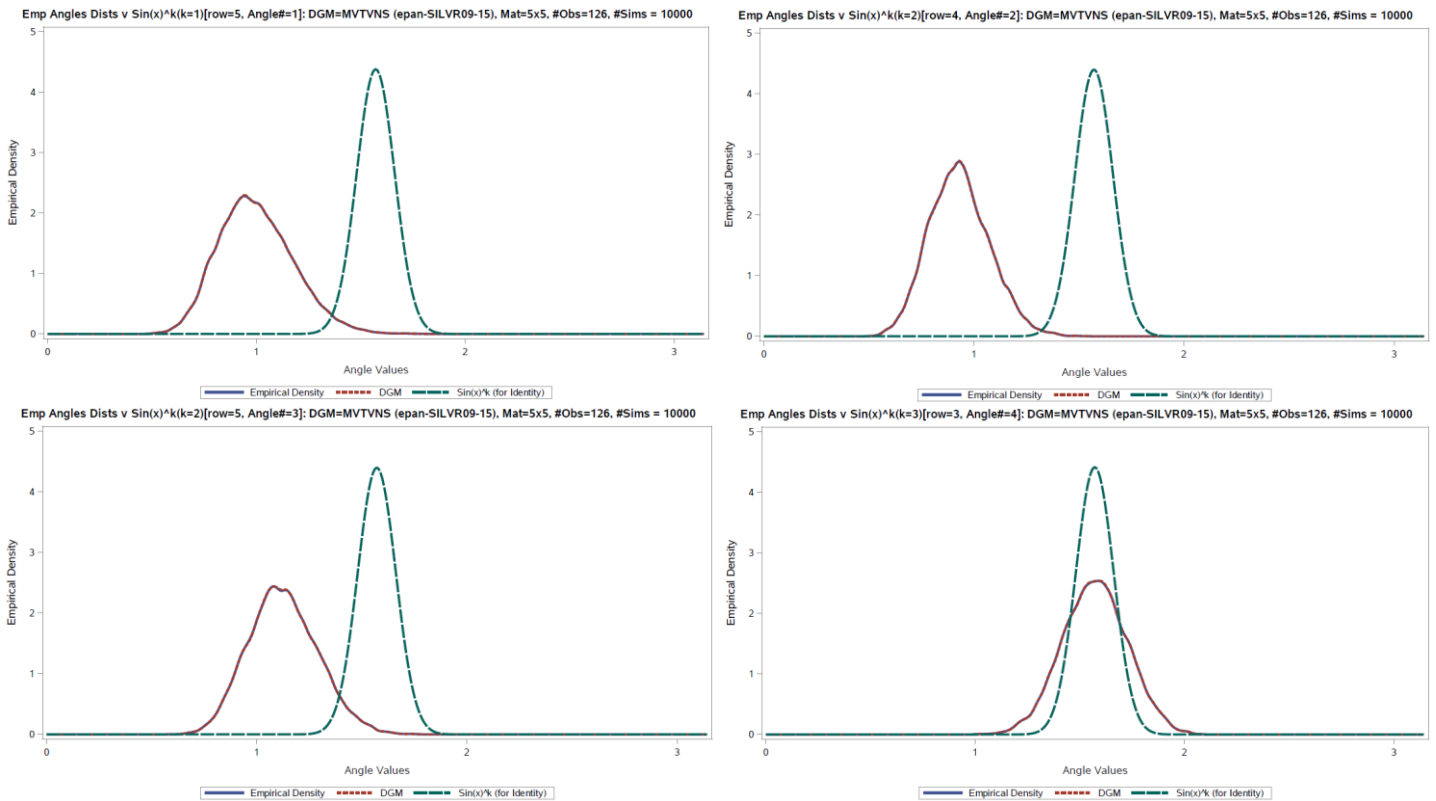
Emp Angles Dists v Sin(x)^k(k=3)[row=4, Angle#=5]: DGM=MVTVNS (epan-SILVR09-15), Mat=5x5, #Obs=126, #Sims = 10000
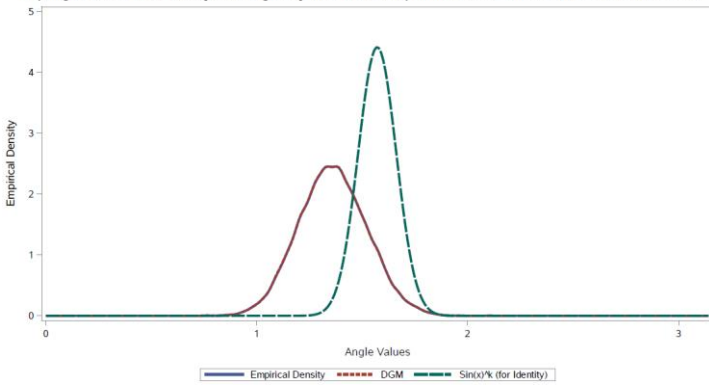
Emp Angles Dists v Sin(x)^k(k=3)[row=5, Angle#=6]: DGM=MVTVNS (epan-SILVR09-15), Mat=5x5, #Obs=126, #Sims = 10000
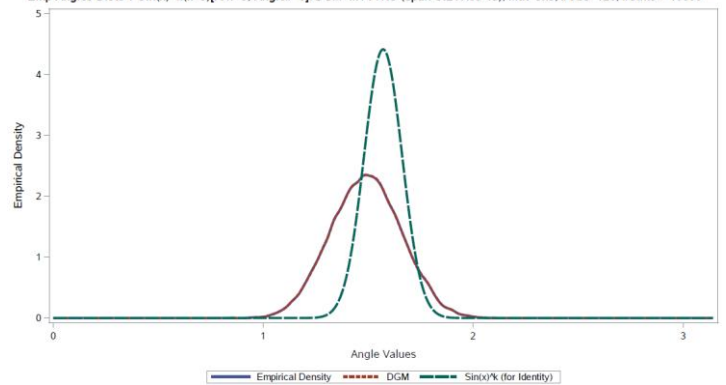
Emp Angles Dists v Sin(x)^k(k=4)[row=2, Angle#=7]: DGM=MVTVNS (epan-SILVR09-15), Mat=5x5, #Obs=126, #Sims = 10000
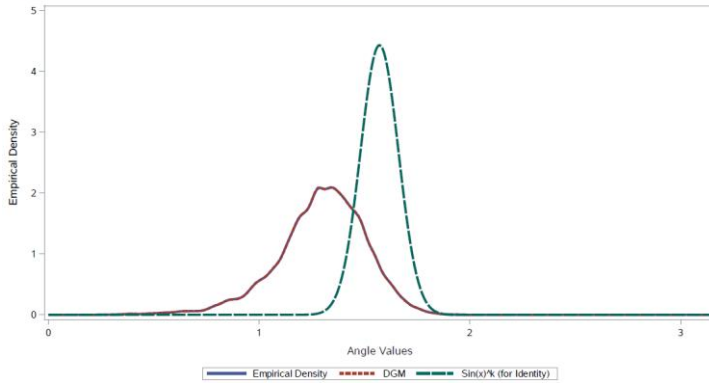
Emp Angles Dists v Sin(x)^k(k=4)[row=3, Angle#=8]: DGM=MVTVNS (epan-SILVR09-15), Mat=5x5, #Obs=126, #Sims = 10000
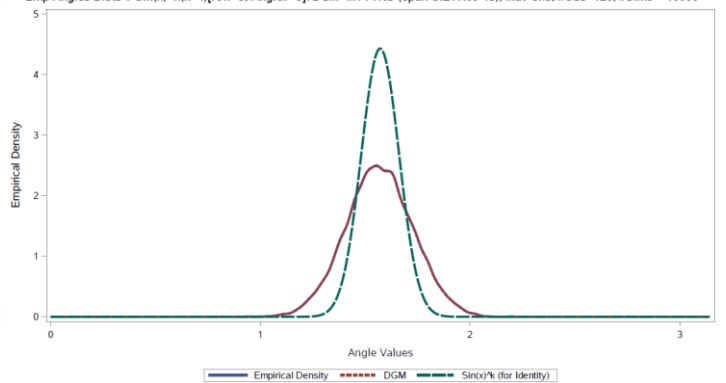
Emp Angles Dists v Sin(x)^k(k=4)[row=4, Angle#=9]: DGM=MVTVNS (epan-SILVR09-15), Mat=5x5, #Obs=126, #Sims = 10000
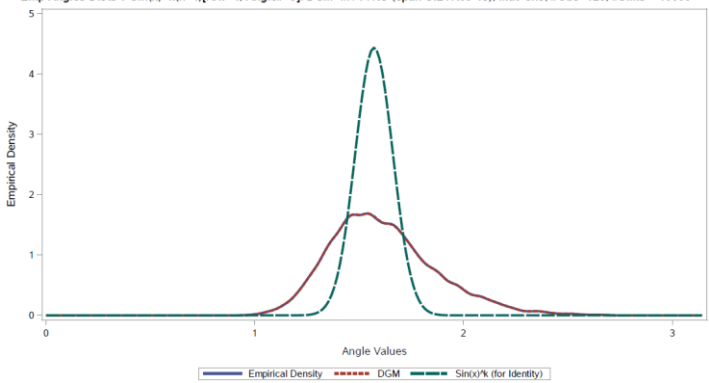
Emp Angles Dists v Sin(x)^k(k=4)[row=5, Angle#=10]: DGM=MVTVNS (epan-SILVR09-15), Mat=5x5, #Obs=126, #Sims = 10000
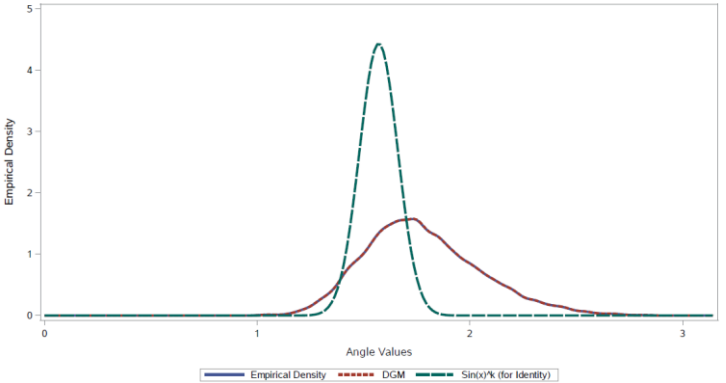
Several points are worth noting and reemphasizing from these graphs. First, the graphs of the angles distributions contain three densities: A. one based on angles perturbation (i.e. sampling from the fitted kernel) as described above in steps 3.-4., B. one based on direct data simulations (steps 1.-2.), and C. the analytical density under the (Gaussian) identity matrix as a comparative baseline. Note that the only reason I include B. is to demonstrate the validity of A., and as expected, the angles distributions from A. and B. are empirically identical (with A. being orders of magnitude faster and more computationally efficient, not to mention inferentially valid). The spectral distributions based on the samples generated in both A. and B. also are identical, as are a wide range of additional aggregated metrics not presented herein (e.g. various norms, VaR-based economic capital, and 'generalized entropy' as in a later chapter below). This empirically validates that the angles-perturbation approach (steps 3.-5.) is an efficient and correct method for isolating and generating the distribution of the correlation/dependence matrix, and unlike steps 1. and 2., one that preserves inferential capabilities. In other words, these results

empirically validate that the angles contain all extant information regarding dependence structure (see Fernandez-Duren & Gregorio-Dominguez, 2023, and Zhang & Yang, 2023, as well as Opdyke, 2024a).

Second, note again that a nonparametric approach works in practice here at least in part because the angles distributions are 'well behaved.' Since they are relatively smooth, typically if not always unimodal, and well bounded, N=10,000 simulations almost always suffice to provide a precise and accurate measure of their distributions. Poorly behaved distributions that are very spikey, multi-modal, and essentially unbounded for all empirical, practical purposes could require numbers of simulations orders of magnitude larger. If N=10,000,000 or even 1,000,000 for example, this approach could be computationally prohibitive in many cases for real-world-sized portfolios, which often exceed p=100 with p(p-1)/2=4,950 pairwise associations/cells.

Finally, as described above, note the multi-modal and high-upper-bounded nature of the spectral distribution for this portfolio compared to the angles distributions, where the biggest thing approaching an estimation challenge is a modest asymmetry. But this speaks only to estimation issues. More notable is the fact that on a cell-by-cell basis, the angles distributions deviate materially i. not only from central values of **π**/2, and less dramatically from perfect symmetry when compared to their (analytic) distributions under the (Gaussian) identity matrix, but also ii. from each other! Each angle's distribution can vary quite notably compared to the other angles' distributions, especially under smaller sample sizes. There simply is no way that one spectral distribution for a matrix, or even p distributions for each eigenvalue individually, even if perfectly estimated, will be able to capture and reflect all the richness of dependence structure reflected here at the granular level of all the p(p-1)/2 pairwise cells. This remains true regardless of their use in this setting, whether for cell-level attribution analyses, granular scenario and reverse scenario analyses, cell-level intervention 'what if' analyses, or customized stress testing, let alone precise and correct inference at either the cell level OR the matrix level. I now leave comparisons to spectral distributions behind[43] to cover implementation issues below.

Nonparametric Kernel Estimation

Due to the bounded nature of the angles distributions on $\theta \in (0, \pi)$, the nonparametric kernel must be appropriately reflected at the boundary (see Silverman, 1986) via:

if $\theta < 0$ then $\theta \leftarrow -\theta$; if $\theta > \pi$ then $\theta \leftarrow (2\pi - \theta)$, which is asymptotically valid. As per the standard implementation, the kernel itself is defined as

(35) $f_h(\theta) = \dfrac{1}{N}\sum_{i=1}^{N} K_h(\theta - \theta_i) = \dfrac{1}{hN}\sum_{i=1}^{N} K_h\big([\theta - \theta_i]/h\big)$ with

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[43] Continued reliance on spectral approaches for this specific problem brings to mind a quotation attributed to John M. Keynes: "the difficulty lies not so much in developing new ideas as in escaping from old ones."

Gaussian: $K(\theta) = \left(1/\sqrt{2\pi}\right) \cdot e^{-\theta^2/2}$, Epanechnikov: $K(\theta) = (3/4) \cdot \left(1 - \theta^2\right), \; |\theta| \leq 1$.

I have tested both the Gaussian and the Epanechnikov kernels extensively in this setting,[44] along with three different bandwidth estimators, h, from Silverman (1986) and one from Hansen (2014), respectively: $h = 1.06 \cdot \hat{\sigma} \cdot N^{-1/5}$, $h = 0.79 \cdot \text{IQR} \cdot N^{-1/5}$, $h = 0.9 \cdot \min\left(\text{IQR}/1.34, \hat{\sigma}\right) \cdot N^{-1/5}$, and lastly, $h = 2.34 \cdot \hat{\sigma} \cdot N^{-1/5}$ for Epanechnikov only, where $\hat{\sigma}$ = sample standard deviation and IQR = sample interquartile range.

As with virtually all kernel implementations, the choice of kernel matters less than the choice of bandwidth, although in this setting, across a broad range of data conditions and correlation/dependence values, the Epanechnikov kernel appears to perform slightly 'better' (i.e. with slightly less variance, thus providing slightly more statistical power) than the Gaussian, perhaps because its sharp bounds require reflection at the boundary less often than the Gaussian kernel (although reflection at the boundary is quite uncommon, even for 'extreme' dependence matrices). The bandwidth that appears to perform best across wide-ranging conditions is $h = 0.9 \cdot \min\left(\text{IQR}/1.34, \hat{\sigma}\right) \cdot N^{-1/5}$. Additionally, for larger matrices (e.g. p=100), bandwidths need to be tightened by multiplying h by a factor of 0.15. When there are many cells (e.g. for p=100, #cells=p(p-1)/2=4,950) this tightening avoids a slight drift in metrics that are aggregated across all the cells (e.g. correlation matrix norms, spectral distributions, and LNP (a type of 'generalized entropy' defined below)). Multiplying by this factor for smaller matrices does not adversely affect the density estimation in any way, so this factor always is used. For matrices much larger than p=100, a further tightening of this factor may be required, and this is readily determined by empirically comparing the distributions of these aggregated metrics under i. direct data simulation (steps 1. and 2.) vs. ii. NAbC's kernel-based sampling (steps 3., 4. and 5.).

Once the kernels have been estimated and the angles distributions generated by perturbing/sampling based on those kernels, the p-values and confidence intervals for both the individual correlation/dependence cells and the entire correlation/dependence matrix are the same as those derived for the Gaussian identity matrix. The only difference, aside from their now-nonparametric basis, is that the angles distributions are no longer symmetric by definition, as is true under the (Gaussian) identity matrix. This can be seen in the Graphs 1-10 of the angles distributions provided above. The p-value calculation, however, remains very straightforward, and it requires just a bit of care to properly account for asymmetry. The one-sided p-value remains simply (36):

(36) one-sided p-value = $F_X(x;k)$ or $1 - F_X(x;k)$ for lower and upper tails, respectively,

where k = n – column# – 2

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[44] Note that the Epanechnikov kernel is used in very closely related problems in this setting (see for example Burda and Jarosz, 2022).

However, due to possible (probable) asymmetry, the two-sided p-value is different, not just two times the one-sied p-value, and requires first the calculation of the empirical mean correlation matrix from the simulations in step 2. of the five sampling steps above. This mean correlation matrix is then translated into a matrix of angles, and we obtain the empirical cdf values corresponding to these "mean angles" with a "look-up" on the entire angles distributions generated in step 4. These cdf's will be close to 0.5 when the angles distributions are close to symmetry, and they will deviate from 0.5 under asymmetry, and will serve as the baseline off of which the two-sided p-values are calculated. Specifically, the difference between the cdf values of each of the angles of the specified correlation matrix being 'tested,' and those of the "mean angles," defines the two-sided p-values, which are simply the sum of the probability in the tails BEYOND this difference. Formulaically this is shown in (37):

(37)   two-sided p-value = max[0, Mcdf – d] + max[0, 1 – (Mcdf + d)], where
$$d = abs(Mcdf – cdf), Mcdf = mean\ angle\ cdf, cdf = cdf\ of\ specified\ angle$$

This usually results in summing both tails, but under notable asymmetry, sometimes only one tail is used. Below is a graphical example of both cases, where the cdf of the "mean angle" is 0.6 and the cdf of the relevant angle in the specified correlation matrix (i.e. the correlation matrix for which we are obtaining p-values) is cdf=0.1 in the single-tail case (Graph 11) and cdf=0.85 in the double-tail case (Graph 12). In the statistical sense, however, both cases remain two-sided p-values.

Graph 11: p-value for a single specified (more) extreme angle cdf
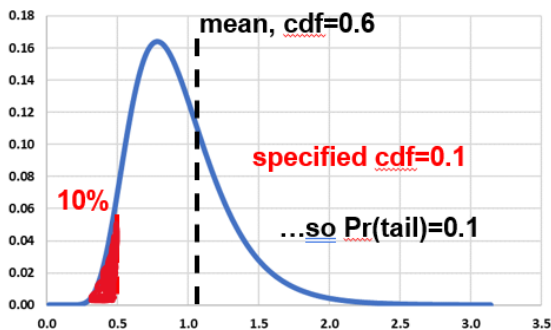
Graph 12: p-value for a single specified non-extreme angle cdf
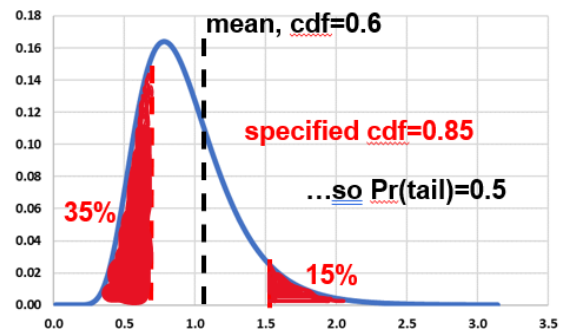
Note that while cdf=0.1 is hardly more 'extreme' than cdf=0.85 in absolute terms, relative to the mean angle cdf=0.6, it is twice as 'extreme,' i.e. twice as far from the mean cdf=0.6 with a distance of 0.5 for Graph 11, but a distance of only 0.25 for Graph 12. Moreover, the tail probability of Graph 11 (0.1) is only 1/5 that of Graph 12 (0.5) (compare the red shaded areas). This example demonstrates why asymmetry must be properly taken into account in this setting, but the two-sided p-value still remains a very straightforward calculation, and the "mean angles" matrix is used for additional, important purposes below, as discussed in the Scenarios section.

Cell-level confidence intervals still are simply calculated as in (31), which automatically takes asymmetry into account as we are using the empirical cdf. This is identical to the same calculation under the (Gaussian) identity matrix (sans the known symmetry of the cdf). And the matrix-level p-value, again, is simply one minus the probability of observing the sample matrix that was observed or one 'less

extreme,' exactly as in (32). This, again, is analogous to the definition of controlling the family-wise error rate (FWER) where all the cells of the matrix comprise the joint null hypothesis. Finally, just as under the (Gaussian) identity matrix, calculation of the confidence interval for the entire matrix remains (33) as previously.

Importantly, again note that we can go in either direction regarding these results: we can specify a correlation/dependence matrix and, under the null hypothesis of the specified correlation matrix, obtain the p-values of an observed matrix, both for the individual cells and the entire matrix, simultaneously. We also have the matrix-level quantile function: we can specify a matrix of cdf values and obtain its corresponding, unique correlation/dependence matrix, which can be extremely useful and straightforward when constructing reverse (stress) scenarios. Finally, we can use simultaneous confidence intervals to obtain the two correlation matrices that form the matrix level confidence interval. An example with all these results is shown in the "One Example" section below, but first I extend NAbC's range of application beyond Pearson's to all dependence measures with positive definite all-pairwise matrices.

NAbC: Any (PD) Dependence Measure, Any Data, Any Matrix Values

To show that NAbC can be applied to any dependence measure with a positive definite all-pairwise matrix, just as it is to Pearson's, we just need to show 1. that the relationship between angles and Pearson's holds not just for Pearson's, but for any positive definite matrix; and 2. the all-pairwise matrices of the dependence measures listed and described in the Background all are positive definite. As shown in (20), (21), and (22), 1. has been proven and is well established in the literature (see Joarder and Ali, 1992, Pinheiro and Bates, 1996; Rebonato and Jackel, 2000; Rapisarda et al., 2007; Pouramadi and Wang, 2015; Cordoba et al., 2018; and Lan et al., 2020). For the "big 3" (Pearson's, Kendall's and Spearman's), 2. has been proven by many, including Sabato et al. (2007). I prove 2. below for all dependence measures with all-pairwise matrices with unit diagonals and values ranging from zero to one on the off-diagonals, i.e. all those discussed in the Background section above. Embrechts et al. (2016) did this already specifically for the tail dependence matrix. Recall the definition of positive definiteness (for a matrix of dimension p):

if $x'Rx > 0$ for all $x \in \mathbb{R}^p \setminus \mathbf{0}$, then R is positive definite.

Because all of the (0,1) dependence measures described above are defined by $0 \le R_{i,j} \le 1$ for all $i \ne j$ and $R_{i,i} = 1$ and $R_{i,j} = R_{j,i}$, $x'Rx$ can be written in quadratic form as

$$(38) \quad x'Rx = \sum_{i=1}^{p} x^2 + 2\sum_{i=1}^{p-1} \sum_{j=i+1}^{p} R_{i,j} x_i x_j$$

As long as $0 < R_{i,j} < 1$ for all $i \ne j$, that is, the coefficients on the cross terms (the second term of (38)) all remain BETWEEN 0 and 1, then

(39) $\sum_{i=1}^{p} x^2 + 2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p} R_{i,j}x_i x_j > 0$ and so $x'Rx > 0$, always, and so R is positive definite.

In the p = 2 case, for example, R is positive definite if $R_{1,1} > 0$ and $\left(R_{1,1}R_{2,2} - R_{1,2}^2\right) > 0$, which is always true when $0 < R_{i,j} < 1$ for all $i \neq j$ and $R_{i,i} = 1$. For the boundary cases, if $R_{i,j} = 0$ for all $i \neq j$, R obviously remains positive definite as the first term of (38) always is greater than zero and the second term disappears; if $R_{i,j} = 1$ for some but not all $i \neq j$ then R is singular; and if $R_{i,j} = 1$ for all $i \neq j$ then R is positive semi-definite, although this case of perfect multivariate dependence is only textbook relevant. In practice, empirically, positive semi-definiteness only is relevant as a boundary condition, as it relates to empirical matrices that approach singularity. Consequently, this means that all dependence measures with values ranging from 0 to 1 are, in practice, positive definite, and that NAbC can be applied to them to define their finite sample distributions. Empirical examples of this are shown in following chapters.

Operationally, implementing NAbC on these (0, 1) measures is no different from implementing it on Pearson's or Kendall's or Spearman's; the (0, 1) instead of (–1, 1) range does not even change how we reflect at the boundary when fitting the nonparametric kernel. This is because specific cells of the Cholesky factor can validly be negative, making the assignation in the last line of the "Correlations to Angles" code in Table A above sometimes assign an angle value slightly above $\pi/2$, even though $\pi/2$ corresponds to a measure value of zero.[45] So this is a soft upper boundary in this case, even though the measure's range of (0,1) typically is not.[46] So when NAbC generates angle θ, we continue to reflect based on: if $\theta < 0$ then $\theta \leftarrow -\theta$; if $\theta > \pi$ then $\theta \leftarrow (2\pi - \theta)$, since for measures with a (0,1) range, the upper bound of π will never be reached, and the lower bound of zero remains valid and hard. So NAbC applies in exactly the same way, for all of these positive definite dependence measures, whether their range of values is (–1, 1) or (0, 1).

Finally, again note that the condition of symmetric positive definiteness holds not only for all relevant dependence measures, as shown above, but also under all relevant real-world data conditions: that is, multivariate financial returns data whose marginal distributions typically are characterized by varying and different degrees of asymmetry, heavy-tailedness, (non-)stationarity, and serial correlation. So this is a very weak and general condition, allowing for the extremely wide-ranging application of NAbC.

---

[45] Note that angle values (which range from zero to π on the hyper-hemisphere) decrease while dependence measure values increase, so a measure value of -1 corresponds to an angle value of π, a measure value of zero corresponds to an angle value of $\pi/2$, and a measure value of 1 corresponds to an angle value of zero (see Zhang et al., 2015 and Lu et al., 2019).

[46] On a related issue, note that Chatterjee's correlation, for example, is bounded by (0,1) only asymptotically, and finite sample results can exceed these bounds. However, when applying NAbC to this and other measures in hundreds of thousands of data simulations under widely varying conditions, as an empirical matter such finite sample exceedences never caused NAbC's angles distributions to deviate from those of direct data simulations, nor did they ever make empirical matrices not positive definite.

I present below graphs of the spectral and angles distributions for some of the dependence measures discussed above, beyond Pearson's, under simulated data reflecting challenging, real-world data conditions (see Opdyke, 2024a, for the application of NAbC to a very wide range of different data conditions). As in the above example, the multivariate returns distribution of the simulated portfolio is generated based on the t-copula of Church (2012), with p=5 assets, varying degrees of heavy-tailedness (df=3, 4, 5, 6, 7), skewness (asymmetry parameter=1, 0.6, 0, -0.6, -1), non-stationarity (standard deviation=3σ, σ/3, σ; 1/3 observations each), and serial correlation (AutoRegressive1=-0.25, 0, 0.25, 0.50, 0.75), with a block correlation structure shown in (34) below and n=126 observations, for half a year of daily returns.[47]

(34)

| 1	-0.3	-0.3	0.2	0.2
-0.3	1	-0.3	0.2	0.2
-0.3	-0.3	1	0.2	0.2
0.2	0.2	0.2	1	0.7
0.2	0.2	0.2	0.7	1

For verification purposes only, I compare those angles distributions based on the data simulation directly against those based on NAbC's kernels, and in all cases the results are empirically indistinguishable. The same is true for the spectral distributions, which I also present below against the Marchenko-Pastur distribution as a(n independence) baseline (see Marchenko and Pastur, 1967). The empirical results yield both expected, and some additional interesting findings.

First, note that the spread, and the spread and shifts, of both the spectral and angles distributions, respectively, are larger for Pearson's than for Kendall's, which is consistent with the former's relative sensitivity to more extreme values under many conditions. The shifts and spread of both measures are much larger than those of Chatterjee,[48] although this is largely due to the fact that while Chatterjee is generally more powerful under dependence that is cyclical or non-monotonic in some way, it is less powerful under associations that are more monotonic, and the data conditions of this example fall more (but not entirely) into the latter category. The story changes a bit when we use the dependence measure suggested by Zhang (2023), which is essentially a maximum between Spearman's rho and Chatterjee's correlation, its objective being to obtain large, if not the maximum power under both types of dependence structures (i.e. strong monotonic dependence as well as cyclical or otherwise non-monotonic dependence). This shows how readily NAbC can be applied to any (positive definite)

---

[47] Note again that this is only approximately Church's (2012) copula, which incorporates varying degrees of freedom (heavy-tailedness) and asymmetry, because I also impose serial correlation and non-stationarity on the data (and subsequently empirically rescale the marginal densities).

[48] The symmetric version of Chatterjee's correlation coefficient is used here (see Chatterjee, 2021), with the finite sample bias correction proposed by Dalitz et. al., 2024.

Graphs 13a: Angles Distributions--NAbC Angles Kernel v Data Simulations v Identity Matrix

Pearson's Rho　　Kendall's Tau　　Chatterjee's　　Spearman's Rho+Chatterjee

Pearson's Rho

Kendall's Tau

Chatterjee's

Spearman's Rho+Chatterjee

Beating the Correlation Breakdown

JD Opdyke, Chief Analytics Officer
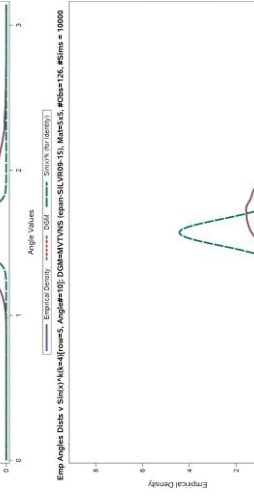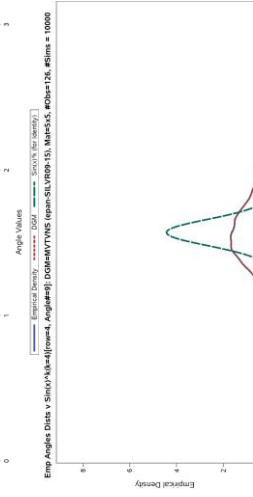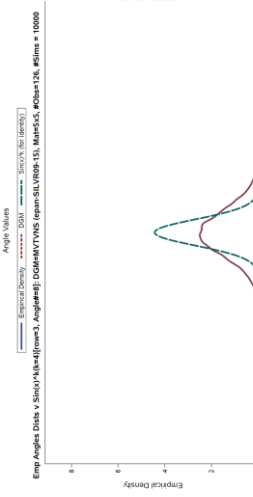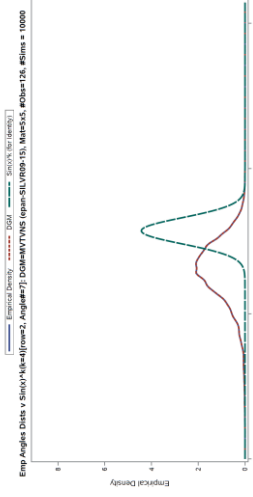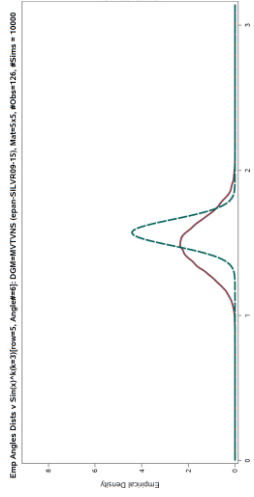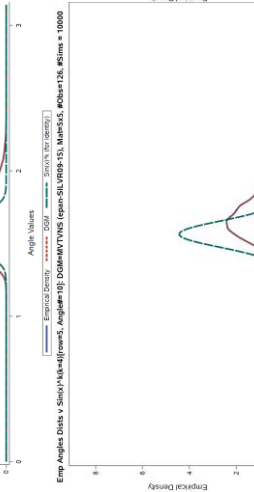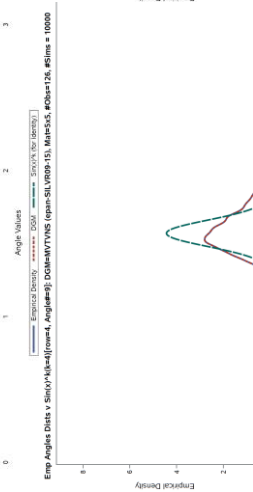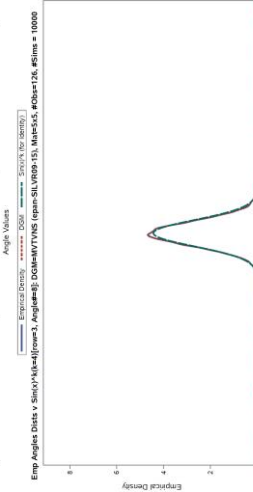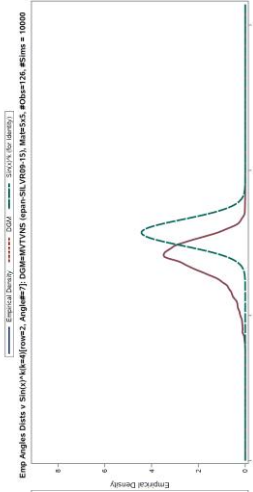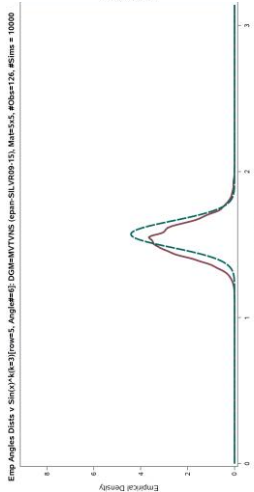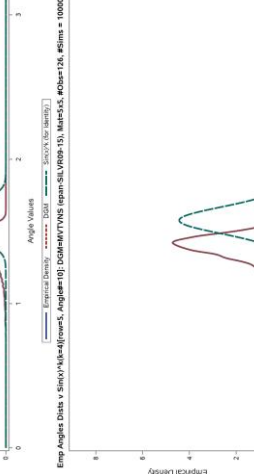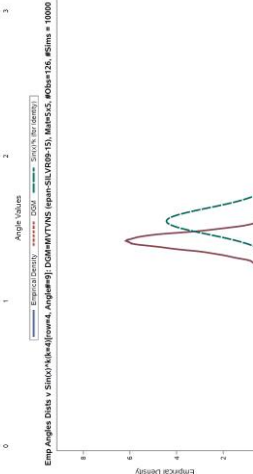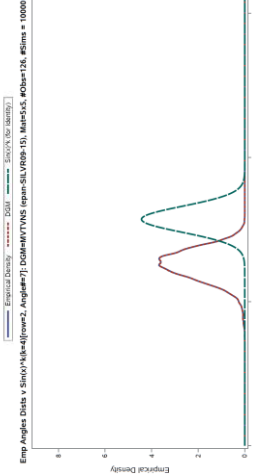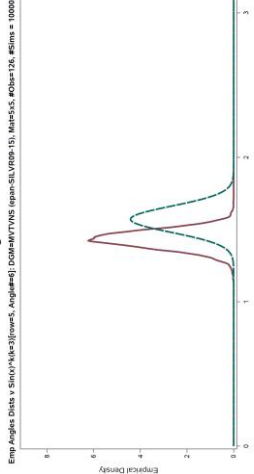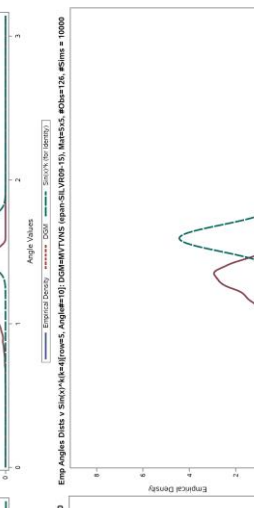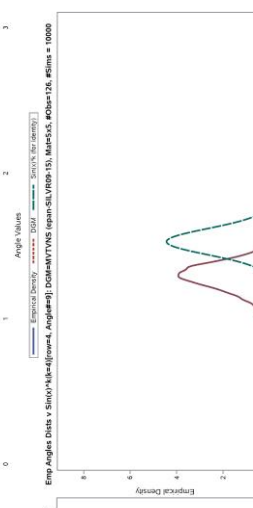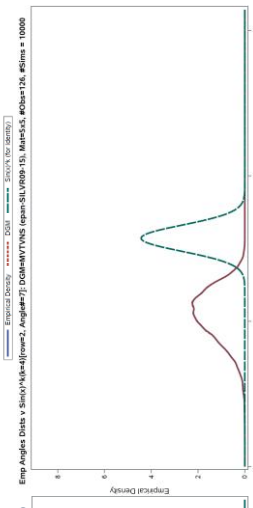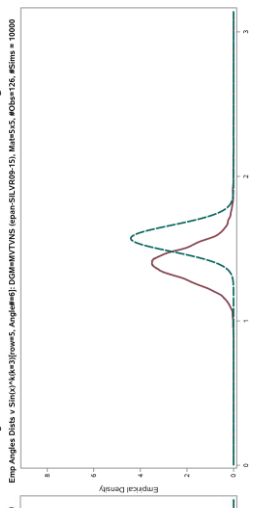
Graph 14: Spectral Distribution-NAbC Angles Kernel v Data Simulations v Marchenko Pastur

Pearson's Rho

Kendall's Tau
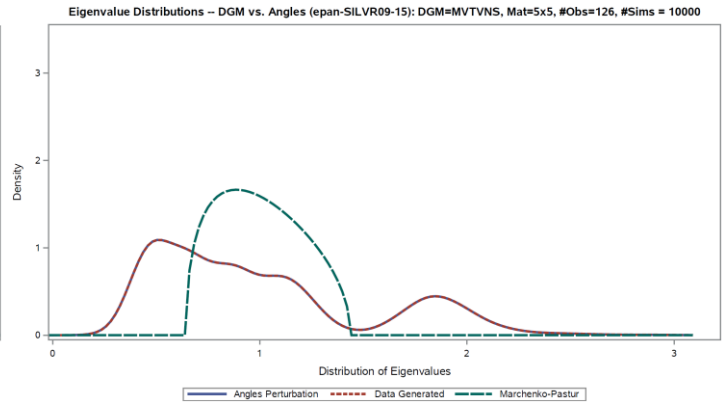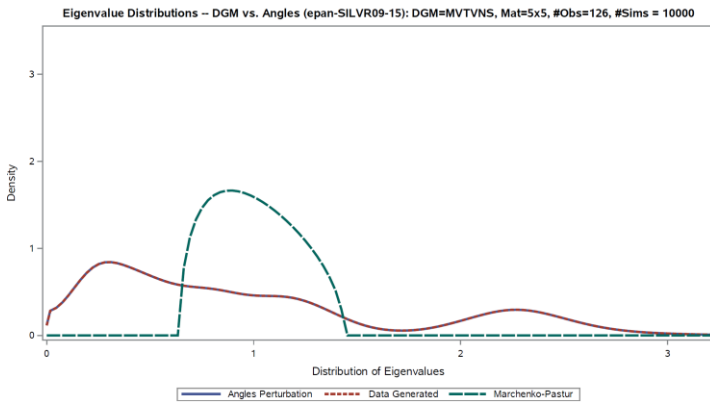
Chatterjee's

Spearman's Rho+Chatterjee
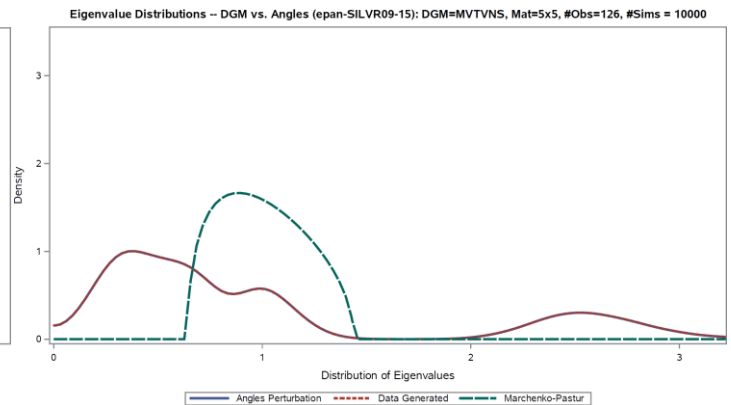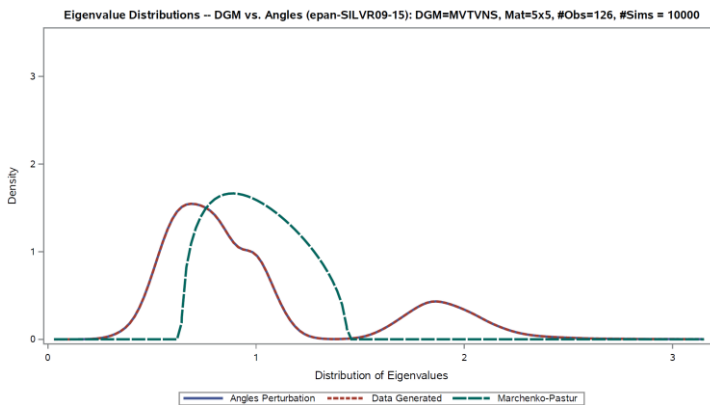
dependence measure, and its utility for making cross-measure comparisons, all else equal, using the same, universally applicable method.

Before providing a complete example of NAbC's application below, i.e. one that provides both matrix and cell level p-values and confidence intervals, and checks all but one of the original objectives listed in the Introduction, I treat two of its additional and important capabilities: the first is its use as a two-sample test of two correlation/dependence matrices, and the second is as a framework for fully flexible scenario analytics, providing granular, realistic scenario analytics far beyond what any of its competitors can provide.

NAbC: Fully General Conditions, Statistical Comparison of Two Matrices

The above development of NAbC's sampling distribution for purposes of statistical inference – generating p-values and confidence intervals at both the cell and matrix levels, with analytical consistancy across levels – has so far covered only hypothesis tests against a matrix of fixed values, i.e. a one-sample test. But we can use NAbC to perform two-sample tests of one sampled matrix against another sampled matrix, say, from two different sectors or two different business lines, where the null hypothesis is no

difference between the dependence structures of the two sectors. The implementation is very similar to the one sample case, except that N=10,000 samples based on the estimates of each of the two angle matrices are generated separately. Then the differences between the two groups of samples of angles, sample i from the first minus sample i from the second, are calculated, and the N differences are tested against the values of the identity matrix, i.e. values of zero representing zero difference between the two matrices, similar to testing a single sample against the null hypothesis of the identity matrix. The only difference is that we must use the "mean angles" cdfs to account for asymmetry in the angles distributions slightly differently: instead of averaging the correlation matrices and then converting this average matrix into the matrix of "mean angles" cdfs, we just calculate the average of the difference-between-angles distributions directly across all the simulations, for each cell, to obtain the "mean angles" cdfs. This avoids incorporating into the asymmetry adjustment what are possibly true differences between the two matrices, i.e. the hypothesis we are testing. The only constraints on this approach are that the two matrices being compared should be the same type of dependence measure (e.g. Szekely's vs. Szekely's, not Szekely's vs. Chatterjee's) and have the same dimension.[49] I include in the "One Example" section below an empirical example of this two-sample application of NAbC. This procedure is readily extended to the multi-sample case, but this is addressed in more detail in future research. Below, I briefly describe how NAbC remains "estimator agnostic" before moving on to NAbC's application to fully flexible scenarios.

NAbC Remains "Estimator Agnostic"

As mentioned briefly in the Introduction, another important and useful characteristic of NAbC is that it remains "estimator agnostic," that is, valid for use with any reasonable estimator of any of the dependence measures being modeled (e.g. Kendall's or Pearson's or Chatterjee's, etc.). Different estimators will have different characteristics under different data conditions. For example, some will provide minimum variance / maximum power, while others may provide unbiasedness or less bias, while others may provide more robustness, and/or different and shifting combinations of these characteristics. Ideally, we would like to be able to use estimators that provide the best trade-offs for our purposes under the conditions most relevant to our given portfolio. Fortunately, NAbC "works" for any estimator, as the relationship between correlations/dependence measure values and angles requires only symmetric positive definiteness. NAbC's finite sample distribution and its resulting inferences obviously will inherit the advantages and disadvantages of the estimator being used, but this is generally an advantage as it

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[49] Note that, as an empirical method, the ability to implement NAbC relies on the degree to which the empirical distributions of the angles approximate continuous distributions over their entire sample spaces. This is largely controlled by the number of simulations run, and fortunately the "good behavior" of the angles distributions renders N=10,000 simulations, which is computationally feasible even for non-small matrices, more than sufficiently large in most cases. However, empirically challenging cases can arise. For example, if we are comparing two sample matrices where some cell values are quite different, the distribution of the difference between the two angles distributions (corresponding to the same cell in each matrix) may not contain the value zero, in which case the empirically-based p-value would be exactly zero. Like any empirical method (e.g. bootstraps, permutation tests, etc.) care must be taken to ensure that the consequences of such results are noted, understood, and properly accounted for.

provides flexibility to use the 'best' estimator under the widest possible range of conditions. Note that all empirical results presented herein use the sample estimators specified in the Background section, and sample sizes in every example all exceed 10p (10 times the dimension of the matrix), which is a widely used threshold for whether a more sophisticated, bias-correcting estimator is needed, at least for Pearson's matrix (see Bongiorno et al., 2023). As mentioned in the Estimation chapter above, I recommend for conditional (forecast) estimation the Average Oracle (AO) of Bongiorno et al. (2023) (see also Bongiorno & Challet, 2023a, for an extensive empirical study against competitors). Further testing may show that AO can be applied to all positive definite dependence measures as well, although this currently is the topic of my continuing research. Now in the next section, I show how all of the other previously derived characteristics of NAbC remain valid for the scenario-restricted case, that is, when selected cells of the all-pairwise matrix are 'frozen' as dictated by specific scenarios, while the rest are allowed to vary.

NAbC: Granular, Fully Flexible Scenarios, Reverse Scenarios, and Stress Testing

"Correlation is one of the most important, if not the most important, risk factor in finance, driving everything ... however, a unified and generally accepted correlation risk management framework does not yet exist" (Packham & Woebbeking, 2023, p.1).

The size, breadth, and surprise of the effects of correlation breakdowns are well documented in the literature (see Kim & Finger, 1998; Loretan & English, 2000; Li et al., 2024; BIS, 2011a, 2011b; Nawroth et al., 2014; Ng et al., 2014; Yu et al., 2014; Chmeilowski, 2014; Epozdemir, 2021; Feng & Zeng, 2022; and Parlatore and Philippon, 2024), if underappreciated during periods of relative market calm: "Furthermore, joint distributions estimated over periods without panics will misestimate the degree of correlation between asset returns during panics." (FRB Chairman, Alan Greenspan,1999). And yet despite its importance, the ability to model, predict, and mitigate correlation breakdowns effectively across very different scenarios, in a fully flexible way, has remained elusive.

To start with, although many approaches do otherwise, it is not enough to stress only the inputs to a correlation/dependence matrix – the matrix itself must be stressed and evaluated under stressed conditions of a particular (extreme) scenario: "... in order to calculate stressed VaR accurately it is also necessary to stress the correlation matrix ... most correlations tend to increase during market crises, asymptotically approaching 1.0 during periods of complete meltdown, such as occurred in 1987, 1998 and 2008. ...Certain methods that could be meaningful [include e]mploying fat-tailed distributions for the risk factors and replacing the standard correlation matrix with a stressed one... ." (BIS, 2011a). Secondly, if a method perturbs eigen decompositions and/or polar angles to obtain correlation/dependence measure distributions, this cannot be done on an ad hoc basis, using mathematically convenient distributions, like the Gaussian, to perturb eigen values, or arbitrary bounded functions, like inverse tangent, to perturb angles (see Galeeva, 2007). Spectral and spherical distributions follow specific and

often known distributions under various conditions, and such approaches need methodological support, whether theoretical or empirical or both, to justify their use when taking what is otherwise a smart approach to generating scenario-specific correlation/dependence matrices. Additionally, such methods must remain cognizant of all the characteristics of the conditions they are attempting to generate. Hardin et al. (2013), for example, utilize a normalized vector of independent gaussian random variables to perturb the observed correlation matrix, but correctly note that "The amount of noise that can be added to the original matrix is determined by its smallest eigenvalue. ... We provide the user with ... a general algorithm to apply to any correlation matrix for which the smallest eigenvalue can be reasonably estimated." (emphasis added). Unfortunately, as mentioned above, this eliminates what are arguably the most widely observed correlation matrices in finance – those based on a 'spiked' covariance matrix (see Johnstone, 2001) where one or a few eigenvalues dominate and the majority of eigenvalues are close to zero, i.e. not reliably estimated. Robustness as dependence matrices approach singularity/NPD is an important quality of any method, but it remains especially critical in financial applications.

Several other approaches avoid these limitations (see Packham and Woebbeking, 2021, Chmielowski, 2014, and Parlatore and Phillippon, 2024) but they have other arguable limitations (e.g. Packham and Woebbeking, 2021, enforce positive definiteness ex post, which as described above distorts the desired distributions of dependence measures), and none provide granular, cell-level control, to restrict perturbation on any combination of cells, while still obtaining a valid distribution of the remaining cells of the correlation/dependence matrix being used. Yet this is exactly what is needed for realistic scenario analytics and stress testing, let alone precise attribution analyses and 'what if' analysis capabilities. Correlation/dependence matrices under a tech market bubble (2000) vs those under a housing bubble (2008) vs those under Covid (2020) will change very different individual cells of the all-pairwise matrix, and very different combinations of cells, in very different ways, often in terms of both direction and magnitude, while leaving many cells strongly affected under one upheaval completely unaffected under another (see Feng & Zeng, 2022). But some combinations of cells will change in similar ways, and distributional analyses must be able to accommodate every possible combination of changes, in terms of both magnitude and direction. In other words, while correlation 'breakdowns' will occur under all of these extreme conditions, the granular nature of all-pairwise matrices ensures that the fundamentally different (and sometimes similar) nature of these breakdowns will be captured and reflected empirically in all related analyses. Although some approaches settle for stretching across several different covariance/correlation matrices, with fixed values, representing several different scenarios (see Parlatore and Phillippon, 2024), this arguably is simply too rigid and discrete and limited for realistic analyses of the *dynamic distributions* of these matrices in a way that remains robust across qualitatively different, and often as yet unobserved (future) breakdowns. Neither does the separate matrix-by-matrix approach allow for flexible, all-else-equal, targeted 'what if' analyses, or granular attribution analyses. If we are to achieve the same level of flexibility in quantitatively modeling dependence matrices that has been attained for the other parameters in the risk and investment models of our portfolios, practitioners and applied researchers must be able to flexibly and realistically model dependence matrices at the most granular level – that of the individual correlation cells – without unnecessary restrictions.

Despite the research on correlation breakdowns listed above, I am aware of only two other limited attempts at this granular level of modeling in the literature (see Saxena et al., 2023, and Veleva, 2017), and both are restricted in notable ways.[50] Fortunately, NAbC allows for specifying ANY combination of cells, within the framework of the all-pairwise matrix, to be 'frozen' at their current values while allowing all the rest to vary, providing full flexibility within this framework.

Several results allow for this full flexibility. First, 1. independence of the angles distributions allows us to vary individual cells. Second, 2. the distributions of individual correlation cells, as well as the distribution of the entire correlation matrix, both remain invariant to the ordering of the rows and columns of the matrix (see Pourahmadi & Wang, 2015, and Lewandowski et al., 2009). Third, 3. based on 1. and 2., we can exploit the simple mechanics of matrix multiplication so that only selected cells of the matrix are affected, and the rest frozen, as required for a given scenario.

To explain 3., I focus only on the lower triangle of the correlation matrices below in Graphs 15-17, since the upper triangle is just its reflection due to symmetry. Note again that using NAbC, we only perturb angles. We never perturb the correlation values directly. We must always convert to angles, perturb the angle values using the fitted kernels, and then translate back to correlation values. In doing so, when multiplying the Cholesky factor by its transpose, $R = BB^T$, changing a given angle cell in matrix B will affect other cells, but only those cells to the right of it in the same row, and those below the diagonal of the corresponding column, as shown graphically for several examples in Graph 15 below.[51]

GRAPH 15: Mechanics of Matrix Multiplication
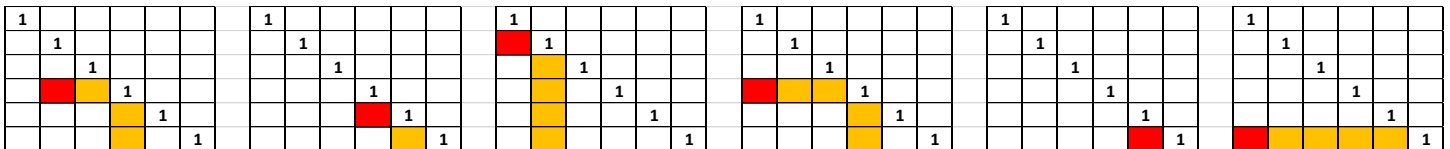
This means that we can simply reorder the matrix so that the targeted cells we want to vary all end up in the rightmost triangle of the lower triangle, according to the fill order in Graph 16 below.

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[50] Saxena et al. (2023) explores the possibility of restricting individual covariance/correlation terms to zero, although they are not always able to enforce this restriction while maintaining positive definiteness. Velena (2017) restricts the values of the correlation matrix being simulated to specified ranges, but only for all off-diagonal cells; in some cases, one algorithm allows for cell-level values to be specified, but without guarantees of positive definiteness.

[51] Note that not all of these (orange) cells will necessarily change if values of zero are involved, but none OTHER than these (orange) cells CAN change when only the red cell changes.

GRAPH 16: Rightmost Triangle Fill Order

Rightmost Triangle Fill Order

11				
12	7			
13	8	4		
14	9	5	2	
15	10	6	3	1

If we only change in matrix *B* the angle values of cells 1, 2, and 3 above, no other cells in the correlation matrix *R* will be affected, simply by virtue of the mechanics of matrix multiplication from $R = BB^T$. Below I show another example. Reorder the correlation matrix so that rows 1-6 are now 6-1 and columns 1-6 are now 6-1, so that the original cells 1,2 and 1,3 and 2,3 and 4,3 are now in the rightmost triangle of the lower triangular matrix, in the fill order shown above.

GRAPH 17: Example of Mechanics of Matrix Multiplication Applied to Rightmost Triangle Fill Order

Determine Targeted Change Cells

1,2				
1,3	2,3			
		4,3		

Reorder Rows/Cols to Fill Rightmost Triangle with Targets According to Fill Order

11				
12	7			
13	8	4		
14	9	5	2	
15	10	6	3	1

Changes in Corresponding Angles Cells ONLY change Same in Resorted Matrix

11				
12	7			
13	8	4,3		
14	9	5	2,3	
15	10	6	1,3	1,2

Changes to the corresponding cells in the angles matrix *B* (the orange cells) will only change these same cells , after $R = BB^T$, in the resulting correlation matrix, *R*, leaving the rest unaffected. Note that the green cells to be targeted for change do not even have to be contiguous, nor do they have to completely 'fill' the rightmost (orange) triangle (note that cells 5 and 6 in the center matrix are not targeted): they only must fill the rightmost triangle according to the order of the center matrix above. Note also that the "rightmost triangle" rule is nested/hierarchical: if I wanted to perform 'what if' analyses on only one of those cells (e.g. cell "1,2") without changing the other three, I order the original correlation matrix to place that cell as the 'first' in the lower triangle of the *B* matrix, as shown. Then, subsequent changes to it will not affect the other (orange) cells, let alone any other non-orange cells. In contrast, changes to cell "4,3" will affect the values of the other orange cells (but not the non-orange cells). Readers are encouraged to test this in the interactive spreadsheet (url link provided above).

So we can exploit these four simultaneous conditions – 1. independence of the angles distributions; 2. (correlation/dependence matrix) distribution invariance to row and column order; 3. the mechanics of matrix multiplication; and 4. the granular, cell-level geometry of NAbC – to obtain great flexibility in defining scenarios wherein some cells vary and some do not. No other approach allows this degree of

flexibility, which is what is required for defining correlation/dependence matrices for use in realistic, plausible, and sometimes extreme stress market scenarios. This also greatly simplifies attribution analyses, isolating and making transparent the identification of effects due to specific pairwise associations, which is something spectral and more aggregated analyses cannot do in this setting.

The above allows for the specification of ANY scenario within the structure of the pairwise matrix. Note, however, that some scenarios can include combinations of cells which are forced to include (in the lower right triangle) one or a few cells not affected by the scenario. This is unavoidable due to the structure of the pairwise matrix: for example, in the matrix above, there are only p! (i.e.5!=120) ways to sort the rows and columns, but there are [p(p1-)/2]! (i.e.15!= 1,307,674,368,000) ways to sort the 15 cells freely. The matrix obviously cannot accommodate freely sorting the individual cells in this way because it breaks the structure of the matrix. Some scenarios, therefore, could conceivably be required to include for perturbation some few additional cells in the lower rightmost triangle that are not relevant to the scenario and otherwise should be held constant. Fortunately, in practice, especially with large matrices, this appears to be a relatively rare occurrence, and when it happens, the effects are identifiable so that materiality can be assessed via 'what if' analyses on the specific cells. But dealing with these potential cases appears to be well worth the price of the unmatched flexibility that using NAbC and the all-pairwise matrix provides,[52] not to mention the other advantages it maintains over more complex, strictly multivariate dependence structures. For usage with actual market data, the latter typically are more difficult to estimate with the same levels of precision and statistical power, let alone to manipulate for purposes of intervention or mitigation. In contrast, pairwise associations are directly identifiable, typically more easily and accurately estimated,[53] and intervention tests are more targeted and transparent.

To conclude this section, I deal with one final implementation issue. When the matrix is scenario-restricted, and we only perturb a subset of the matrix while keeping the remaining cells fixed, what values do we use for those 'frozen' cells? This is where the mean angles matrix, used to account for asymmetry when calculating the two-sided p-values in the previous section, comes into play. When the matrix angles are sampled using the fitted kernel densities, a sample is drawn from the entire matrix, and if it is scenario restricted, the sampled values for those cells that are 'frozen' are simply overwritten with their means. So after N=10,000 samples, all 10,000 values of the 'frozen' cells unaffected by the scenario will

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[52] Most of the related scenario literature perturbs scenario-based cells and simply ignores their (often notable) effects on the rest of the matrix (which should remain 'frozen,' but isn't), not to mention the effects of the rest of the matrix on the scenario-related cells. These papers euphemistically refer to the former as 'peripheral' correlations (see Ng et al. (2014) and Yu et al. (2014)). NAbC is the only method that fully controls the values and thus, the indirect effects of these so-called 'peripheral' correlations.

[53] They also can be *estimated* rigorously, and with targeted precision and flexibility, with well-established methods such as vine copulas (see Czado and Nagler, 2022)). Ironically, however, when used for *inference or sampling* for this problem specifically, vine copulas and similar methods become extremely unwieldy and much more complex and less transparent than NAbC, not to mention ungeneralizable beyond Pearson's (see the vine and extended onion algorithms of Lewandowski et al. (2009), and the similar chordal sparsity method of Kurowicka (2014)).

have the same mean value for that specific cell, and when translated via $R = BB^T$ back into correlation matrices, all the correlation values for those cells will be their mean correlation value. In other words, their values will not change, and will remain 'frozen,' based on a reasonably robust estimator of their true value (note that these 'frozen' values are not based on a single estimated matrix, but rather, they each are the means of N=10,000 matrices). The order of magnitude of empirical accuracy of these values is inversely related to the number of samples drawn, N. In the example in the "One Example" section below, we observe accuracy to the fourth decimal place for these frozen cells when N=25,000 simulations, as expected. Alternately, the values could be treated as truly known constants from the beginning, but it is more conservative (and realistic) to use estimates based on the mean of all the samples.[54]

I end this section by reemphasizing that this matrix sorting method for providing fully flexible scenarios, within the framework of the all-pairwise matrix, applies not only to Pearson's, but also to all positive definite dependence measures, under the fully general data conditions for financial portfolios described above. One complete, empirical example of this, and all of NAbC's applications, covering all but one of its original objectives described in the Introduction, is shown below.

NAbC Example: Kendall's Tau p-values & Confidence Intervals, Unrestricted & Scenario-restricted

Now, with NAbC's characteristics and broad range of application described above, I can present a complete example of its implementation here.[55] This example will check seven of the eight original objectives boxes listed in the Introduction above (solely for ease of results replication, the data generating mechanism for this example is simply multivariate standard normal, but the example 'works' when a 'real world' data generating mechanism is used). The dependence measure chosen is Kendall's Tau, under two cases: unrestricted, and scenario-restricted. NAbC provides both p-values and confidence intervals, at both the cell level and matrix level, with N=25k simulations and number of observations n = 160, representing about eight months of daily market returns. The values of the matrix [A] are based on a Pearson's matrix from A' below, translated to A via $\tau = (2/\pi)\arcsin(r)$, which is valid

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[54] Note that when NAbC is being used as a two sample test in the scenario-restricted setting, we are only testing, by design, the scenario-relevant cells. So the mean values of each matrix are inserted into the 'frozen' cells of each matrix just as in the one-sample test, but then those cells are ignored thereafter, i.e. when calculating cell-level p-values, as well as the (restricted) matrix-level p-values, just as in the one-sample test. This ensures that the two-sample test is conducted only for the scenario-relevant cells, even as we properly perturb each entire matrix when generating the samples.

[55] See Opdyke (2024a) for extensive additional examples under wide ranging data conditions.

under elliptical data (see McNeil et al., 2005),[56] and approximately valid under some broader classes of distributions (see Hansen & Luo (2024) and Hamed (2011) for examples).

[A'] =

1				
0.2	1			
-0.1	0.3	1		
0.3	-0.3	-0.1	1	
0.6	0.4	0.0	0.1	1

UNRESTRICTED CASE: Given a specified or well-estimated dependence matrix [A], and its specified or well-estimated data generating mechanism:

[A]

1				
0.13	1			
-0.06	0.19	1		
0.19	-0.19	-0.06	1	
0.41	0.26	0.00	0.06	1

[B]

0.8				
0.7	0.8			
0.8	0.7	0.7		
0.7	0.8	0.8	0.7	

[C]

| 1				
0.3	1			
0.1	0.1	1		
0.05	-0.1	0.1	1	
0.5	0.25	0.2	0.15	1

Q1. **Confidence Intervals**: What are the two dependence matrices that correspond to the lower– and upper–bounds of the 95% confidence interval for [A]? What are, simultaneously, the individual 95% confidence intervals for each and every cell of [A]?

Q2. **Quantile Function**: What is the unique dependence matrix associated with [B], a matrix of cumulative distribution function values associated with the corresponding cells of [A]?

Q3. **p-values**: Under the null hypothesis that observed dependence matrix [C] was sampled from the data generating mechanism of [A], what is the p-value associated with [C]? And simultaneously, what are the individual p-values associated with each and every cell of [C]?

Q4. **p-values**: Under the null hypothesis that observed dependence matrix [A] and observed dependence matrix [C] each were sampled from the same population, and therefore have the same values, what is the matrix-level p-value? And simultaneously, what are the individual p-values associated with each and every cell of the matrix?

SCENARIO-RESTRICTED CASE: Under a specific scenario only selected pairwise dependence cells of [A] will vary (green), while the rest (red) are held constant, unaffected by the scenario (e.g. COVID). This is

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[56] See Koike et al. (2024) for a sophisticated paper defining the conditions under which Pearson's retains the invariance property under marginal transformations.

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matrix [D].

[D]	**[E]**	**[F]**

1				
0.13	1			
-0.06	0.19	1		
0.19	-0.19	-0.06	1	
0.41	0.26	0.00	0.06	1

	0.8			
	0.8	0.8	0.7	

1				
0.13	1			
-0.06	0.350	1		
0.19	-0.19	-0.06	1	
0.41	0.180	0.125	0.215	1

Q5. **Confidence Intervals**: What are the two dependence matrices that correspond to the lower– and upper–bounds of the 95% confidence interval for [D] (holding constant the non-selected red cells)? What are, simultaneously, the individual 95% confidence intervals for only those cells of [D] that are relevant to the scenario (green)?

Q6. **Quantile Function**: What is the unique dependence matrix associated with [E], a matrix of cumulative distribution function values associated with the corresponding (green) cells of [D]?

Q7. **p-values**: Under the null hypothesis that observed dependence matrix [F] was sampled from the (scenario-restricted) data generating mechanism of [D], what is the p-value associated with [F] (with red cells held constant)? And simultaneously, what are the individual p-values associated with every (non-constant, green) cell of [F]?

Q8. **p-values**: Under the null hypothesis that observed dependence matrix [D] and observed dependence matrix [F] each were sampled from the same population, and therefore have the same values (for their unrestricted green cells), what is the matrix-level p-value (for the unrestricted portion of the matrix)? And simultaneously, what are the individual p-values associated with each and every (green, unrestricted) cell of the matrix?

Answers to these questions require inference at both the cell and matrix levels, simultaneously and with cross-level consistency, as well as requiring the matrix-level quantile function, all under both the unrestricted and scenario-restricted cases, under any data conditions. Only NAbC can simultaneously answer Q1.-Q8. above under general data conditions, as shown below in Tables B1 and B2.

For Q1 and Q5, the two top matrices correspond to the first (matrix-level) question, and the bottom two matrices correspond to the second (cell-level) question. Note the wider intervals on a cell-by-cell basis for the matrix-level confidence intervals compared to the cell-level confidence intervals, as expected. Also note, for Q3 and Q7, the smaller p-values for the individual cells compared to the respective matrix-level p-values, which are larger, as expected, as they are analogous to family-wise error rate (FWER) of a joint hypothesis covering all cells of the matrix. Note also that the green cells of Q6 differ from the corresponding cells in Q2: even though the (green) angles distributions themselves remain unaffected by scenario restrictions, the ultimate correlation values of those cells ARE affected due to the matrix multiplication of the Cholesky factor, $R = BB^T$. Comparing the two-sample test of Q4 to the one-sample test of Q3, we find, as expected, the increased variability from two samples increases all the cell-level p-values, as well as the matrix-level p-value. Only when we double the sample size (as well as the number

TABLE B1: NAbC Provides Complete Inference, Example of Kendall's Tau – Cell and Matrix Level p-values and Confidence Intervals

Q1

1				
-0.0172	1			
-0.2100	0.0626	1		
0.0472	-0.3567	-0.1602	1	
0.2794	0.0926	-0.1873	-0.0830	1

1				
0.0250	1			
-0.1661	0.0986	1		
0.0926	-0.3131	-0.1427	1	
0.3210	0.1410	-0.1396	-0.0478	1

Q2

1				
0.1729	1			
-0.0355	0.2369	1		
0.2374	-0.1510	-0.0392	1	
0.4335	0.3159	0.0614	0.1040	1

1				
0.2735	1			
0.0910	0.3475	1		
0.3323	0.0127	0.1182	1	
0.5250	0.4370	0.2335	0.2789	1

1				
0.2300	1			
0.0424	0.3013	1		
0.2920	-0.0525	0.0570	1	
0.4929	0.3849	0.1611	0.2103	1

Q3

p-value=0.1503

1				
0.0006	1			
0.0090	0.0218	1		
0.0222	0.0315	0.0269	1	
0.0170	0.0157	0.0088	0.0077	1

Q4

p-value=0.5370

1				
0.0213	1			
0.0342	0.0723	1		
0.0593	0.1350	0.1148	1	
0.1194	0.0922	0.0043	0.0789	1

p-value=0.1285
(n=320, N=50k)

1				
0.0042	1			
0.0078	0.0072	1		
0.0056	0.0319	0.0263	1	
0.0282	0.0110	0.0008	0.0131	1

Q8

p-value=0.4436

1				
	1			
	0.0328	1		
			1	
0.3992	0.0290	0.0140	1	

p-value=0.2251
(n=320, N=50k)

1				
	1			
	0.0017	1		
			1	
0.2231	0.0008	0.0001	1	

Q7

p-value=0.0436

1				
	1			
	0.0047	1		
			1	
0.0077	0.0148	0.0171	1	

Q6

1				
0.1282	1			
-0.0636	0.2398	1		
0.1942	-0.1940	-0.0639	1	
0.4098	0.2895	0.0362	0.0867	1

1				
0.1282	1			
-0.0636	0.3492	1		
0.1942	-0.1940	-0.0639	1	
0.4098	0.3425	0.1150	0.1841	1

1				
0.1282	1			
-0.0636	0.3006	1		
0.1942	-0.1940	-0.0639	1	
0.4098	0.3165	0.0809	0.1486	1

Q5

1				
0.1282	1			
-0.0636	0.0478	1		
0.1942	-0.1940	-0.0639	1	
0.4098	0.1757	-0.1144	-0.0541	1

1				
0.1282	1			
-0.0636	0.0904	1		
0.1942	-0.1940	-0.0639	1	
0.4098	0.2028	-0.0789	-0.0184	1

of simulations to cautiously and accurately account for smaller p-values) do we obtain similarly small p-values of the same order of magnitude. Similar patterns hold for the one-sample vs two-sample test results under the scenario-restricted cases (Q7 and Q8, respectively). Finally, note that the empirical values of the red cells in Q5-Q6 differ slightly from those in [D] and [F]. This is due to NAbC's conservative use of the mean of the estimated angles (correlation) matrices, rather than presuming we know the absolute 'true' values of these cells (although this is justified in some specific cases).

In terms of actual runtimes, note that NAbC is somewhat computationally intensive, but not prohibitively so. Implementing NAbC on synthetic data representing real-world data conditions (e.g. margins with different and varying degrees of asymmetry, (non)stationarity, serial correlation, and heavy-tailedness) for non-small portfolios of dimension 100x100, on a six-year old commodity laptop with RAM=32GB but no multi-threading, NAbC generates a full set of results, based on N samples = 10,000, in about 2.4 hours. However, in a multithreaded environment, let alone one with more memory, NAbC could be applied on similarly non-small matrices in minutes. For the specific case of the gaussian identity matrix, applying inverse probability transform sampling as described above, on a 100x100 matrix with N samples = 10,000, NAbC takes less than 25 minutes to run on the same laptop. Notably, Roman (2023) benchmarked NAbC's sampling under the gaussian identity matrix against that of Makalic & Schmidt (2018) and obtained up to a 30% reduction in runtime under NAbC. But of course, NAbC's analytic solution under these conditions is instantaneous (see url for excel workbook above). So while NAbC is not a result that currently can be used "real-time" for, say, high frequency trading (except for when the fully analytic solution is valid), its runtimes remain reasonable given its very generalized application and widely available modern computing resources.

NAbC: Beyond 'Distance'— LNP, A Generalized Entropy

In a relevant and validating digression, it is intriguing and important to note that the (two-sided) cell-level p-values NAbC provides (see Q3 and Q7 in Table B above) actually can be used to construct a competitor to commonly used distance metrics, such as norms, and it has a number of advantages over them in this setting. Some commonly used norms for measuring correlation 'distances' include the Taxi, Frobenius/Euclidean, and Chebyshev norms (collectively, the Minkowski norm), shown below in (40).

(40) $$\|x\| = \left(\sum_{i=1}^{d} |x_i|^m \right)^{1/m}$$ where x is a distance from a presumed or baseline correlation value, d=number of observations, and m=1, 2, and ∞ correspond to the Taxi, Frobenius/Euclidean, and Chebyshev norms, respectively.

All of these norms measure absolute distance from a presumed or baseline correlation/dependence value. But the range of all relevant and widely used dependence measures is bounded, either from –1 to 1 or 0 to 1, and the relative impact and meaning of a given distance at the boundaries are not the same as those in the middle of the range. In other words, a shift of 0.02 from an original or presumed correlation/dependence value of, say, 0.97, means something very different than the same shift from

0.37. NAbC's p-values attribute probabilistic MEANING to these two different cases, while a norm would treat them identically, even though they very likely indicate what are very different events of very different relative magnitudes with potentially very different consequences.

Therefore, a natural, PROBABILISTIC distance measure based directly on NAbC's cell-level p-values is the natural log of the product of the p-values, dubbed 'LNP' in (41) below:

$$(41) \quad \text{"LNP"} = \ln\left(\prod_{i=1}^{q} p\text{-}value_i\right) = \sum_{i=1}^{q} \ln\left[p\text{-}value_i\right] \text{ where } q = p(p-1)/2 \text{ and } p\text{-}value_i \text{ is 2-sided.}$$

Using a Pearson's correlation matrix under the (Gaussian) identity matrix, LNP shows a very strong correspondence with the entropy of the correlation matrix, defined by Felippe et al. (2021 and 2023) as (42) below:

$$(42) \quad \text{Entropy} = Ent(R/p) = -\sum_{j=1}^{p} \lambda_j \ln(\lambda_j)$$

where R is the sample correlation matrix and λ_j are the p eigenvalues of the correlation matrix after it is scaled by its dimension, R/p. Importantly, this result (42), like NAbC, is valid for ANY positive definite measure of dependence, not just Pearson's. Graph 18 below compares LNP of Kendall's Tau matrix to the entropy of Kendall's Tau matrix in 10,000 simulations (with n=126 for half a year of daily returns) under the Gaussian identity matrix, and the Pearson's correlation between them (0.98) is virtually identical to the same comparison based on Pearson's rather than Kendall's (just under 0.99). [57]

It is important to note, however, that entropy here is limited to being calculated relative to the case of independence, which for many dependence measures corresponds only with the identity matrix.[58] In contrast, LNP can be calculated, and retains its meaning, in all cases, based on ANY values of the dependence matrix, not just the case of independence. Yet the correspondence of LNP to entropy under this specific case speaks to LNP's natural interpretation as a meaningful measure of deviation/ distance/ disorder (depending on your interpretation), and one that also is more flexible and granular than entropy as it is measured cell-by-cell, p(p-1)/2 times, as opposed to only p times for p eigenvalues. As such, LNP might be considered a type of 'generalized entropy' relative to any baseline of the dependence measure, as specified by the researcher, that is not necessarily perfect (in)dependence. Such entropy-related measures certainly are relevant in this setting as entropy has been used increasingly in the literature to measure, monitor, and analyze financial markets (see Meucci, 2010b, Almog and Shmueli, 2019,

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[57] In addition, the Pearson's correlation between LNP and the entropy of Felippe et al. (2021 and 2023), under these conditions of the Gaussian identity matrix, was the same – 0.98 – for both Spearman's and Chatterjee's.

[58] Recall, of course, that a zero value for Pearson's or Kendall's or Spearman's does not imply independence, but independence does imply a zero value for these measures.

Graph 18: Identity Matrix Simulations for Kendall's Tau – LNP v Matrix Entropy
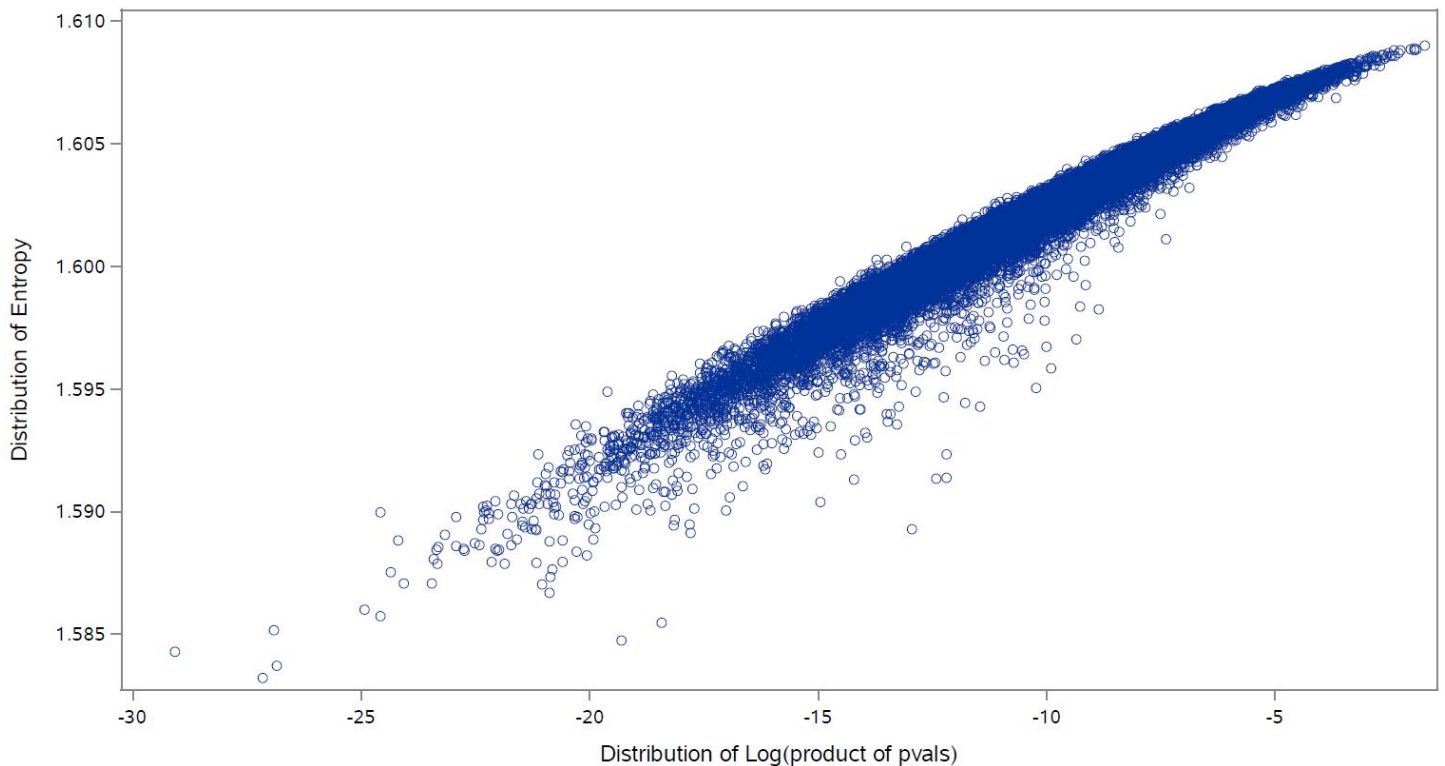
Chakraborti et al., 2020, and Vorobets, 2024, 2025, for several examples). So the use of LNP here warrants further investigation as a matrix-level measure that, unlike widely used distance measures such as norms, has a solid and meaningful probabilistic foundation. Its calculation applies not only beyond the independence case generally, but also to ALL positive definite measures of dependence, regardless of their values. LNP's range of application is as wide as that of NAbC's matrix-level p-value, and the two are readily calculated side-by-side as they are both based on NAbC's cell-level (two-sided) p-values for the entire matrix. These are intriguing results with possibly far-reaching implications.

NAbC: Future Research and Applications

There are a number of areas where additional research can further validate and potentially increase the utility and/or breadth of application of NAbC.

Analytic Angles Distributions: I provide above the derivation of NAbC's fully analytic solution under the Gaussian identity matrix, but this is a narrow (albeit foundational) case. Although NAbC's general solution remains 'runtime reasonable' for its purposes and generality, expanding the range of conditions for an analytic solution for the angles distributions would dramatically speed NAbC's implementation. Deriving an "all cases" analytic solution currently appears to be a nontrivial problem, but even providing this under additional, specific cases would be useful and directly useable in NAbC's application.

Competing Distributional Methods: Implementing and comparing NAbC's results to those of competing, if less flexible methods, like Hansen & Archakov (2021) and the Bayesian approaches of Lan et al. (2020) and Ghosh et al. (2020), likely would be useful and insightful exercises, especially if the focus is on power studies and tests of robustness under common dependence structures in finance (e.g. spiked covariance matrices and otherwise near-singular matrices). The same goes for the two-sample case, where NAbC compares two sample matrices against the null hypothesis of no difference between them: comparing NAbC's results against those from some of the purportedly more generalizable competitors, like Ding et al. (2023) (after covariances are converted to matrices of Pearson's correlations) and Wang et al. (2025), would add to our knowledgebase.

Statistical Process Control: A full implementation of NAbC within a statistical process control (SPC) monitoring framework would be useful to compare against potential competitors like those of Adegoke et al. (2022), Ajadi et al. (2021), Bours & Steland (2020), Wang et al. (2019), Choi and Shin (2021), and those reviewed in Ebadi et al. (2021). While a major focus should be on power-related metrics like average run length, special scrutiny should be placed on robustness and the (nonparametric) generalizability of NAbC vs these competitors, since these characteristics arguably are areas of weakness in the SPC literature, and where NAbC might make its most meaningful contributions.

Recovering DAGs: Causal model frameworks typically are defined, in part or in whole, by directed acyclic graphs (DAGs), and the recovery of the 'ground truth' DAG, assuming it is rightly specified,[59] is one of causal modeling's fundamental tasks.[60] While NAbC obviously is not designed to provide "all else equal" estimates of the magnitudes of treatment effects that regression approaches within causal frameworks can provide (see MacKinnon & Lamp, 2022), it may be able to enhance covariate classification efforts for accurate DAG recovery. For example, when using a directional dependence measure, say, Chatterjee's improved correlation (see Xia et al., 2024), we can apply NAbC twice, once with the treatment variable (X) and dependent variable (Y) and relevant covariates (V1, V2, V3) in one order in the matrix (e.g. X, V1, V2, V3, Y), and once in the reverse order in the matrix (Y, V3, V2, V1, X). The two resulting matrices will together capture all potential associations, in both directions, of all the variables. And all the cells of the two estimated dependence matrices will fully map to the relevant causal categories that make up a DAG (e.g. the confounders, colliders, mediators, independent variables, causes of X, consequences of X, causes of Y, and consequences of Y).

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[59] "The correct causal model is an exacting qualification, requiring a program of research with precise definition of causal effects, specification of assumptions, and sensitivity analysis for how violating assumptions affects results. Statistical analysis is useful for demonstrating associations between variables that are consistent or inconsistent with a causal model." (MacKinnon & Lamp, 2022).

[60] Note that Czado (2025) demonstrates that vine copulas, described above as being a very flexible and effective method for *estimating* dependence structure under real-world conditions (if not for *inference* regarding all-pairwise matrices), also can be remarkably effective in the causal discovery setting. See also the innovative causal modeling approaches of Rodriguez Dominguez & Yadav (2024), and Rodriguez Dominguez (2024).

What NAbC could provide here is two things: first, p-values associated with each of these categories, for each variable, to assist in their classification. Unlike an approach that would analyze each variable pair separately, NAbC's p-values would properly take into account the entire dependence matrix, with all the pairwise relationships, simultaneously. Secondly, many asymmetric dependence measures are not readily useable within regression frameworks, even when such frameworks are appropriately directional (see MacKinnon & Lamp, 2022; however, see Marqui et al., 2024, for an exception). For example, I am not aware of any regression, directional or otherwise, that allows for the use of Zhang's (2023) combined correlation or the asymmetric tail dependence measure of Deidda et al, (2023) when estimating (directional) covariate effects.[61] Yet these directional dependence measures may have more power under certain data conditions for identifying, and thus classifying, these relationships, and thus, when used alongside causal models, could enhance their power for accurate DAG recovery. To reemphasize, this is not a proposal for a standalone causal model, but rather, a possible way that NAbC could be used to augment the accurate DAG recovery provided by an existing causal model framework. But of course, this begs the bigger question of whether DAGs can be used reliably within "self-referencing open systems like capital markets" to begin with (Polakow et al., 2023). Importantly, many express strong caution, based on recent and rigorous research, regarding its application in this setting (see de Lara, 2023; Gong et al., 2024).[62] I propose only that NAbC can play an effective role here if the answer to this question turns out to be "yes" or "under some conditions."

Conclusions

NAbC defines the finite sample distributions of an extremely broad range of dependence measures – all those whose pairwise matrices are positive definite – under challenging, real-world financial data conditions. This enables robust inference and ceteris paribus analyses where none before were possible. Motivation for its development has been the need for a method that satisfies all eight of the objectives

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[61] However, Andu et al. (2021) take a very interesting approach using adaptive elastic net regression wherein Szekely's (2007) distance correlation is used to weight parameter estimates in the L1 penalty term of the regression. What's more, Pascual-Marqui et al. (2024) combine their multivariate distance-based Chatterjee correlation with the regression approach of Blömbaum et al. (2019) to extend and robustify association-based results to causal results, thus supporting the utility of using such measures in the causal modeling setting.

[62] From Polakow et al. (2023): "The clarion call for causal reduction in the study of capital markets is intensifying. However, in self-referencing and open systems such as capital markets, the idea of unidirectional causation (if applicable) may be limiting at best, and unstable or fallacious at worst." Polakow et al. (2023). From Gong et al. (2024): "... potential outcomes (PO) and structural causal models (SCMs) stand as the predominant frameworks. However, these frameworks face notable challenges in practically modeling counterfactuals ... we identify an inherent model capacity limitation, termed as the 'degenerative counterfactual problem', emerging from the consistency rule that is the cornerstone of both frameworks." And from De Lara (2024): "Most of the literature on causality considers the structural framework of Pearl and the potential-outcomes framework of Neyman and Rubin to be formally equivalent, and therefore interchangeably uses the do-notation and the potential-outcome subscript notation to write counterfactual outcomes. In this paper, we ... prove that structural counterfactual outcomes and potential outcomes do not coincide in general – not even in law." See Opdyke (2024b) for a more complete review of this literature.

listed below, because to date, no extant method has addressed all of these "real-world necessary" requirements simultaneously. Yet anything less than this, when modeling dependence structure in our risk and investment portfolios, fails to rise to the same level of analytical rigor as has been applied to the other parameters of these models: that is indefensible given that, as recognized in the literature, its effects can be larger than many, if not all of the other parameters combined. I list again the eight objectives below for the reader's convenience:

1. NAbC remains valid under challenging, real-world data conditions, with marginal asset distributions characterized by notably different and varying degrees of serial correlation, (non-)stationarity, heavy-tailedness, and asymmetry

2. NAbC can be applied to ANY positive definite dependence measure

3. NAbC remains "estimator agnostic," that is, valid regardless of the sample-based estimator used to estimate any of the above-mentioned dependence measures

4. NAbC provides valid confidence intervals and p-values at both the matrix level and the pairwise cell level, with analytic consistency between these two levels (i.e. the confidence intervals for all the cells define that of the entire matrix, and the same is true for the p-values; this effectively facilitates, and in many cases makes possible, attribution analyses)

5. NAbC provides valid confidence intervals and p-values not only for one-sample tests against matrices of fixed, assumed 'true' values, but also for two-sample tests comparing two matrices, so that we can assess inferentially whether dependence structures truly are different across different sectors or segments of our businesses.

6. NAbC provides a one-to-one quantile function, translating a matrix of all the cells' cdf values to a (unique) correlation/dependence measure matrix, and back again, enabling precision in reverse scenarios and stress testing

7. all the above results remain valid even when selected cells in the matrix are 'frozen' for a given scenario or stress test – that is, unaffected by the scenario – thus enabling flexible, granular, and realistic scenarios

8. NAbC remains valid not just asymptotically, i.e. for sample sizes presumed to be infinitely large, but rather, for the specific sample sizes we have in reality (for full-rank matrices with n>p), enabling reliable application in actual, real-world, non-textbook settings

For the fundamental but narrow case of Pearson's correlation under the Gaussian identity matrix, I derive NAbC's fully analytic solution, with p-values and confidence intervals at both the cell and matrix levels (along with a measure of generalized entropy), provided in an interactive spreadsheet.

http://www.datamineit.com/JD%20Opdyke--The%20Correlation%20Matrix-Analytically%20Derived%20Inference%20Under%20the%20Gaussian%20Identity%20Matrix--02-18-24.xlsx

But way beyond Pearson's, the fully general NAbC solution presented herein checks all of the eight objectives boxes above, simultaneously. The list of critically important, applied research that NAbC now facilitates, if not makes possible, is not only expansive, but also feasible with an ease of use and interpretability, broad range of application, scalability, and robustness not found in other more limited (spectral) methods with narrow ranges of application. It even appears that NAbC can increase the power of causal models when recovering DAGs under some conditions, further expanding its already comprehensive scope.

With NAbC, we now have a powerful, applied research tool enabling the treatment an extremely broad class of ubiquitous dependence measures with the same level of analytical rigor as the other major parameters in our financial portfolio models. We can use NAbC in frameworks that identify, probabilistically measure and monitor, and even anticipate critically important events, such as correlation breakdowns, and mitigate and manage their effects. Correlation breakdowns are widely documented, arguably endemic characteristics of major financial markets, and their destructive potential on our attempts to estimate and forecast market behavior is difficult to overestimate. Modeling efforts in this area simply cannot be effective without knowledge of, and the ability to implement and utilize, the true sampling distributions of the relevant dependence measures under real world conditions. In providing exactly these distributions, in a useable, transparent, and straightforward way, NAbC should prove to be a very useful means by which we can better understand, predict, and manage portfolios in our multivariate world.

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